

CHARACTERIZATIONS OF PRODUCT HARDY SPACES IN BESSEL SETTING

XUAN THINH DUONG, JI LI, BRETT D. WICK, AND DONGYONG YANG

ABSTRACT. In this paper, we work in the setting of Bessel operators and Bessel Laplace equations studied by Weinstein, Huber, and the harmonic function theory in this setting introduced by Muckenhoupt–Stein, especially the generalised Cauchy–Riemann equations and the conjugate harmonic functions. We provide the equivalent characterizations of product Hardy spaces associated with Bessel operators in terms of the Bessel Riesz transforms, non-tangential and radial maximal functions defined via Poisson and heat semigroups, based on the atomic decomposition, the extension of Merryfield’s result which connects the product non-tangential maximal function and area function, and on the grand maximal function technique which connects the product non-tangential and radial maximal function. We then obtain directly the decomposition of the product BMO space associated with Bessel operators. These results are a first extension for product Hardy and BMO associated to a differential operator other than the Laplacian and are a major step beyond the Chang–Fefferman setting.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

1.1. Background and main results. Multiparameter harmonic analysis was introduced in the ’70s and studied extensively in the ’80s, led by S.-Y. A. Chang, R. Fefferman, R. Gundy, J. Journé, J. Pipher, E. Stein and others (see for example [Cha, GS, CF1, F1, CF2, FSt, CF3, J1, M, F2, F3, J2, P, F4, F5]). The theory of multiparameter harmonic analysis is largely influenced by the corresponding theory of one-parameter (classical) harmonic analysis, but is strongly motivated by two different geometric phenomena. First, one naturally encounters families of rectangles and operators that are invariant under different scalings than the standard one (e.g, the operator is invariant under a scaling in each variable separately and not just a uniform scaling of all the variables). Second, the boundary behavior of analytic functions in several complex variables necessitated an understanding of approach regions that behaved differently in each variable separately. Both these naturally lead to the theory of harmonic analysis allowing for a decomposition of functions admitting different, independent behavior in each variable separately.

As in classical harmonic analysis a key ingredient in the theory is the development of the Hardy and BMO spaces, their duality and the connections to atomic decompositions. In the multiparameter setting the geometry alluded to above leads to a more complicated description of product BMO. As demonstrated by Carleson the natural BMO condition on rectangles is not sufficient to characterize the dual of the Hardy space. This necessitates a BMO theory based on arbitrary open sets and leads to numerous geometric challenges in the theory. An important result in the area is Journé’s covering lemma which provides a tool by which the general open sets can be replaced by certain families of rectangles with controlled geometry. After this important ground work was established, the development of multiparameter harmonic analysis followed the lines of obtaining $T1$ theorems, characterizations of Hardy spaces via non-tangential and radial maximal functions, and Hilbert (Riesz) transforms. More recent developments in the area include the dyadic structures, characterization of product BMO via commutators (see for example [FS, FL, LPPW, LT, LPPW2, Tre, DO, LPW, OPS, KLPW]), and the developments

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of $T1$ and Tb theorems (see for example [HyM, Ou]). As is well known, the product Hardy space H^1 has a variety of equivalent norms, in terms of square functions, maximal functions and Hilbert transforms (Riesz transforms in higher dimension), see for example [L, p. 19]. See also the product Hardy spaces and boundedness of product singular integrals in different versions studied in [HY, CYZ, BLYZ, LBY, LBYZ].

Spaces of homogeneous type were first introduced by R. Coifman and G. Weiss [CW] in the 70's in order to extend the theory of Calderón–Zygmund operators to a more general setting. There are no translations or dilations, and no analogue of the Fourier transform or convolution operation on such spaces. Using Coifman's idea on the decomposition of the identity operator, G. David, J. Journé, and S. Semmes [DJS] developed Littlewood–Paley analysis on spaces of homogeneous type and used it to give a proof of the $T1$ theorem on this general setting. Recently, based on this Littlewood–Paley analysis, Han, the second author, and Lu [HLLu] developed the product Hardy spaces $H^p(X_1 \times X_2)$ for $p \leq 1$ and close to 1 on product spaces of homogeneous type $X_1 \times X_2$ via Littlewood–Paley square function, and proved the duality of H^p with the Carleson measure type spaces CMO^p , see also the related results in [HLL] and [HLW]. Later, the boundedness of singular integrals, the product $T1$ theorem, and atomic decomposition of H^p were also studied in [HLLW, LW, HLLin, HLPW].

The theory of the classical Hardy space is intimately connected to the Laplacian; changing the differential operator introduces new challenges and directions to explore. In the past 10 years, a theory of Hardy spaces associated to operators was introduced and developed by P. Auscher, the first author, S. Hofmann, A. McIntosh, L. Yan and many others (we refer to [DY2, DY3, HM, HLMMY] and the references therein). In [DSTY] they first introduced the product Hardy space on the Euclidean setting associated with operators via area functions, and the product BMO space via Carleson measures and proved the duality. We also refer to [DLY, STY] for the product Hardy spaces on the Euclidean setting associated with operators for the atomic decomposition. Recently, P. Chen, L. Ward, L. Yan and the first and second authors [CDLWY] developed the product Hardy spaces $H_{L_1, L_2}^1(X_1 \times X_2)$ on $X_1 \times X_2$ (the product spaces of homogeneous type) associated with operators via Littlewood–Paley area functions and atomic decompositions, and studied the boundedness of product singular integrals with non-smooth kernels, the Calderón–Zygmund decomposition and interpolations of H^p , as well as the boundedness of Marcinkiewicz type multipliers. Here L_1 and L_2 are two non-negative self-adjoint operators acting on $L^2(X_1)$ and $L^2(X_2)$, respectively, and satisfying Davies–Gaffney estimates. However, the weak conditions on L_1 and L_2 seem not strong enough for obtaining the characterizations of product space $H_{L_1, L_2}^1(X_1 \times X_2)$ via maximal function or via the “Riesz transforms” and the decomposition of product BMO space in this setting is not known either.

In 1965, B. Muckenhoupt and E. Stein in [MSt] introduced the harmonic function theory associated with Bessel operator Δ_λ , defined by setting for suitable functions f ,

$$\Delta_\lambda f(x) := \frac{d^2}{dx^2} f(x) + \frac{2\lambda}{x} \frac{d}{dx} f(x), \quad \lambda > 0, \quad x \in \mathbb{R}_+ := (0, \infty).$$

The related elliptic partial differential equation is the following “singular Laplace equation”

$$(1.1) \quad \Delta_{t,x}(u) := \partial_t^2 u + \partial_x^2 u + \frac{2\lambda}{x} \partial_x u = 0$$

studied by A. Weinstein [W], and A. Huber [Hu] in higher dimension, where they considered the generalised axially symmetric potentials, and obtained the properties of the solutions of this equation, such as the extension, the uniqueness theorem, and the boundary value problem for certain domains.

If u is a solution of (1.1) then u is said to be λ -harmonic. The function u and the conjugate of u (denoted by v) satisfy the following Cauchy–Riemann type equations

$$(1.2) \quad \partial_x u = -\partial_t v \quad \text{and} \quad \partial_t u = \partial_x v + \frac{2\lambda}{x} v \quad \text{in } \mathbb{R}_+ \times \mathbb{R}_+.$$

In [MSt] they developed a theory in the setting of Δ_λ which parallels the classical one associated to the standard Laplacian, where results on $L^p(\mathbb{R}_+, dm_\lambda)$ -boundedness of conjugate functions and fractional integrals associated with Δ_λ were obtained for $p \in [1, \infty)$ and $dm_\lambda(x) := x^{2\lambda} dx$.

We also point out that Haimo [H] studied the Hankel convolution transforms $\varphi_\#^\lambda f$ associated with the Hankel transform in the Bessel setting systematically, which provides a parallel theory to the classical convolution and Fourier transforms. It is well-known that the Poisson integral of f studied in [MSt] is the Hankel convolution of Poisson kernel with f , see [BDT].

Since then, many problems based on the Bessel context were studied, such as the boundedness of Bessel Riesz transform, Littlewood–Paley functions, Hardy and BMO spaces associated with Bessel operators, A_p weights associated with Bessel operators (see, for example, [K, AK, BFBMT, V, BFS, BHN, BCFR, YY, DLWY, DLMWY] and the references therein).

The aim of this paper is to focus on this specific Bessel setting, to study the equivalent characterizations of product Hardy spaces and the decomposition of product BMO spaces associated with Bessel operator Δ_λ .

We note that the measure dm_λ related to Δ_λ is a doubling measure, and hence the standard product Hardy spaces via Littlewood–Paley area functions and via atoms fall into the line of [HLLu, HLPW], see Section 2. Also, the kernels of the Poisson and heat semigroups of Δ_λ satisfy the size, smoothness and conservation property, and hence the product Hardy spaces associated with Δ_λ via Littlewood–Paley area functions and via atoms fall into the line of [CDLWY]. The first part of this paper is to prove that these two versions of Hardy spaces coincide in this Bessel setting, denoted by $H_{\Delta_\lambda}^p$, $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$. We also provide the equivalent characterization via Littlewood–Paley g -functions. Based on this, we obtain that the dual of $H_{\Delta_\lambda}^p$ is the standard Carleson measure spaces studied in [HLLu], denoted by $\text{CMO}_{\Delta_\lambda}^p$ (by $\text{BMO}_{\Delta_\lambda}$ when $p = 1$).

Then the second part, which is the main contribution of this paper, is to provide the equivalent characterizations of $H_{\Delta_\lambda}^p$ in terms of non-tangential and radial maximal function defined via the Poisson and heat semigroups, as well as the characterization via Bessel Riesz transforms. To obtain this, we build up a variant of the technical lemma of K. Merryfield [M], which connects the product non-tangential maximal function and the area function, and we make good use of the generalised Cauchy–Riemann type equations (1.2) and establish the grand maximal function which connects the non-tangential and radial maximal functions. Then, as a direct consequence, we obtain the decomposition of $\text{BMO}_{\Delta_\lambda}$ via the Bessel Riesz transforms. We note that these results in the second part are first extensions for product Hardy and BMO spaces beyond the Chang–Fefferman setting on Euclidean spaces.

1.2. Statement of main results. Throughout the paper, for every interval $I \subset \mathbb{R}_+$, we denote it by $I := I(x, t) := (x - t, x + t) \cap \mathbb{R}_+$. The measure of I is defined as $m_\lambda(I(x, t)) := \int_{I(x, t)} x^{2\lambda} dx$. In the product setting $\mathbb{R}_+ \times \mathbb{R}_+$, we define $d\mu_\lambda(x_1, x_2) := dm_\lambda(x_1) \times dm_\lambda(x_2)$ and $\mathbb{R}_\lambda := (\mathbb{R}_+ \times \mathbb{R}_+, d\mu_\lambda(x_1, x_2))$. We work with the domain $(\mathbb{R}_+ \times \mathbb{R}_+) \times (\mathbb{R}_+ \times \mathbb{R}_+)$ and its distinguished boundary $\mathbb{R}_+ \times \mathbb{R}_+$. For $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, denote by $\Gamma(x)$ the product cone $\Gamma(x) := \Gamma_1(x_1) \times \Gamma_2(x_2)$, where $\Gamma_i(x_i) := \{(y_i, t_i) \in \mathbb{R}_+ \times \mathbb{R}_+ : |x_i - y_i| < t_i\}$ for $i := 1, 2$.

We now provide several definitions of $H_{\Delta_\lambda}^p$, $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$. These spaces all end up being the same, which is one of the main results in this paper. This requires some additional notation, but the careful reader will notice that the spaces are distinguished notationally by a subscript to remind how they are defined.

Following [CDLWY], we define the product Hardy spaces associated with the Bessel operator Δ_λ using the Littlewood–Paley area functions and square functions via the semigroups $\{T_t\}_{t>0}$, where $\{T_t\}_{t>0}$ can be the Poisson semigroup $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$ or the heat semigroup $\{e^{-t\Delta_\lambda}\}_{t>0}$.

Given a function f on $L^2(\mathbb{R}_\lambda)$, the Littlewood–Paley area function $Sf(x)$, $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, associated with the operator Δ_λ is defined as

$$(1.3) \quad Sf(x) := \left(\iint_{\Gamma(x)} \left| t_1 \partial_{t_1} T_{t_1} t_2 \partial_{t_2} T_{t_2} f(y_1, y_2) \right|^2 \frac{d\mu_\lambda(y_1, y_2) dt_1 dt_2}{t_1 m_\lambda(I(x_1, t_1)) t_2 m_\lambda(I(x_2, t_2))} \right)^{\frac{1}{2}}.$$

The square function $g(f)(x)$, $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$, associated with the operator Δ_λ is defined as

$$(1.4) \quad g(f)(x) := \left(\int_0^\infty \int_0^\infty \left| t_1 \partial_{t_1} T_{t_1} t_2 \partial_{t_2} T_{t_2} f(x_1, x_2) \right|^2 \frac{dt_1 dt_2}{t_1 t_2} \right)^{\frac{1}{2}}.$$

We now define the product Hardy space $H_{\Delta_\lambda}^p$ by using (1.3) via Poisson semigroup as follows.

Definition 1.1. For $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, the Hardy space $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ is defined as the completion of

$$\{f \in L^2(\mathbb{R}_\lambda) : \|Sf\|_{L^p(\mathbb{R}_\lambda)} < \infty\}$$

with respect to the norm (quasi-norm) $\|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)} := \|Sf\|_{L^p(\mathbb{R}_\lambda)}$, where Sf is defined by (1.3) with $T_t := e^{-t\sqrt{\Delta_\lambda}}$.

Our first main result is to show that $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ coincide with $H^p(\mathbb{R}_\lambda)$ as in [HLLu]; see Definition 2.5 below.

Theorem 1.2. Let $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$. The space $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ coincides with the classical product Hardy space $H^p(\mathbb{R}_\lambda)$ and they have equivalent norms (or quasi-norms).

As a direct consequence, we have that the dual of $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, denoted by $\text{CMO}_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, is the classical product Carleson measure space $\text{CMO}^p(\mathbb{R}_\lambda)$, which is introduced in [HLLu] (see the precise definition in Section 2). Especially, for $p = 1$, we denote the dual of $H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)$ by $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$.

Remark 1.3. We note that we can also define the product Hardy spaces $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ as in Definition 1.1 using the area function Sf via $T_t := e^{-t\Delta_\lambda}$, as well as using the square function $g(f)$ as in (1.4) via $T_t := e^{-t\sqrt{\Delta_\lambda}}$ or $T_t := e^{-t\Delta_\lambda}$, denoted by $H_{\Delta_\lambda, 1}^p(\mathbb{R}_\lambda)$, $H_{\Delta_\lambda, 2}^p(\mathbb{R}_\lambda)$ and $H_{\Delta_\lambda, 3}^p(\mathbb{R}_\lambda)$, respectively. These three versions of product Hardy spaces coincide with $H^p(\mathbb{R}_\lambda)$ and they have equivalent norms (or quasi-norms). See Proposition 3.2 below.

We now define another version of the Littlewood–Paley area function. Let

$$\nabla_{t_1, y_1} := (\partial_{t_1}, \partial_{y_1}), \quad \nabla_{t_2, y_2} := (\partial_{t_2}, \partial_{y_2}).$$

Then the Littlewood–Paley area function $S_u f(x)$ for $f \in L^2(\mathbb{R}_\lambda)$, $x := (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$ is defined as

$$(1.5) \quad S_u f(x) := \left(\iint_{\Gamma(x)} \left| \nabla_{t_1, y_1} e^{-t_1 \sqrt{\Delta_\lambda}} \nabla_{t_2, y_2} e^{-t_2 \sqrt{\Delta_\lambda}} (f)(y_1, y_2) \right|^2 \frac{t_1 t_2 d\mu_\lambda(y_1, y_2) dt_1 dt_2}{m_\lambda(I(x_1, t_1)) m_\lambda(I(x_2, t_2))} \right)^{\frac{1}{2}}.$$

Then naturally we have the following definition of the product Hardy space via the Littlewood–Paley area function $S_u f$.

Definition 1.4. For $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, the Hardy space $H_{S_u}^p(\mathbb{R}_\lambda)$ is defined as the completion of

$$\{f \in L^2(\mathbb{R}_\lambda) : \|S_u f\|_{L^p(\mathbb{R}_\lambda)} < \infty\}$$

with respect to the norm (quasi-norm) $\|f\|_{H_{S_u}^p(\mathbb{R}_\lambda)} := \|S_u f\|_{L^p(\mathbb{R}_\lambda)}$.

Next we define the product non-tangential and radial maximal functions via heat semigroup and Poisson semigroup associated to Δ_λ , respectively. For all $\alpha \in (0, \infty)$, $p \in [1, \infty)$, $f \in L^p(\mathbb{R}_\lambda)$ and $x_1, x_2 \in \mathbb{R}_+$, let

$$\begin{aligned}\mathcal{N}_h^\alpha f(x_1, x_2) &:= \sup_{\substack{|y_1 - x_1| < \alpha t_1 \\ |y_2 - x_2| < \alpha t_2}} \left| e^{-t_1 \Delta_\lambda} e^{-t_2 \Delta_\lambda} f(y_1, y_2) \right|, \\ \mathcal{N}_P^\alpha f(x_1, x_2) &:= \sup_{\substack{|y_1 - x_1| < \alpha t_1 \\ |y_2 - x_2| < \alpha t_2}} \left| e^{-t_1 \sqrt{\Delta_\lambda}} e^{-t_2 \sqrt{\Delta_\lambda}} f(y_1, y_2) \right|\end{aligned}$$

be the product non-tangential maximal functions with aperture α via the heat semigroup and Poisson semigroup associated to Δ_λ , respectively. Denote $\mathcal{N}_h^1 f$ by $\mathcal{N}_h f$ and $\mathcal{N}_P^1 f$ by $\mathcal{N}_P f$. Moreover let

$$\begin{aligned}\mathcal{R}_h f(x_1, x_2) &:= \sup_{t_1 > 0, t_2 > 0} \left| e^{-t_1 \Delta_\lambda} e^{-t_2 \Delta_\lambda} f(x_1, x_2) \right|, \\ \mathcal{R}_P f(x_1, x_2) &:= \sup_{t_1 > 0, t_2 > 0} \left| e^{-t_1 \sqrt{\Delta_\lambda}} e^{-t_2 \sqrt{\Delta_\lambda}} f(x_1, x_2) \right|\end{aligned}$$

be the product radial maximal functions via the heat semigroup and Poisson semigroup associated to Δ_λ , respectively.

Definition 1.5. The Hardy space $H_{\mathcal{M}}^p(\mathbb{R}_\lambda)$, $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, associated to the maximal function $\mathcal{M}f$ is defined as the completion of the set

$$\{f \in L^2(\mathbb{R}_\lambda) : \|\mathcal{M}f\|_{L^p(\mathbb{R}_\lambda)} < \infty\}$$

with the norm (quasi-norm) $\|f\|_{H_{\mathcal{M}}^p(\mathbb{R}_\lambda)} := \|\mathcal{M}f\|_{L^p(\mathbb{R}_\lambda)}$. Here $\mathcal{M}f$ is one of the following maximal functions: $\mathcal{N}_h f$, $\mathcal{N}_P f$, $\mathcal{R}_h f$ and $\mathcal{R}_P f$.

Based on our first main result Theorem 1.2, the second main result of this paper is as follows.

Theorem 1.6. Let $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$. The product Hardy spaces $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, $H_{S_u}^p(\mathbb{R}_\lambda)$, $H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda)$, $H_{\mathcal{R}_h}^p(\mathbb{R}_\lambda)$, $H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda)$ and $H_{\mathcal{N}_P}^p(\mathbb{R}_\lambda)$ coincide and have equivalent norms (or quasi-norms).

Next we consider the definition of product Hardy space via the Bessel Riesz transforms $R_{\Delta_\lambda, 1}(f)$ and $R_{\Delta_\lambda, 2}(f)$ on the first and second variable, respectively. For the definition of Bessel Riesz transforms, we refer to (2.4) in Section 2.2.

Definition 1.7. The product Hardy space $H_{Riesz}^1(\mathbb{R}_\lambda)$ is defined as the completion of

$$\{f \in L^1(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda) : R_{\Delta_\lambda, 1}f, R_{\Delta_\lambda, 2}f, R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}f \in L^1(\mathbb{R}_\lambda)\}$$

endowed with the norm

$$\|f\|_{H_{Riesz}^1(\mathbb{R}_\lambda)} := \|f\|_{L^1(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda, 1}f\|_{L^1(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda, 2}f\|_{L^1(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda, 1}R_{\Delta_\lambda, 2}f\|_{L^1(\mathbb{R}_\lambda)}.$$

Then based on our result in Theorem 1.6, the third main result of the paper is the following characterization of $H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)$.

Theorem 1.8. The product Hardy spaces $H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)$ and $H_{Riesz}^1(\mathbb{R}_\lambda)$ are equivalent.

Moreover, we also characterize $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ for $p \in ((2\lambda + 1)/(2\lambda + 2), 1)$ via Bessel Riesz transform in a slightly different form. A distribution $f \in (\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ is said to be restricted at infinity, if for any $r > 0$ large enough, $e^{-t_1 \sqrt{\Delta_\lambda}} e^{-t_2 \sqrt{\Delta_\lambda}} f \in L^r(\mathbb{R}_\lambda)$ (for the notation and details of this distribution space, we refer to Definition 2.3 below). By Theorem 1.6 and an argument as in [St, pp. 100-101], we see that for any $f \in H^p(\mathbb{R}_+, dm_\lambda)$ with $p \in ((2\lambda + 1)/(2\lambda + 2), 1]$, $e^{-t_1 \sqrt{\Delta_\lambda}} e^{-t_2 \sqrt{\Delta_\lambda}} f \in L^r(\mathbb{R}_\lambda)$ for all $r \in [p, \infty]$.

Theorem 1.9. *Let $p \in ((2\lambda + 1)/(2\lambda + 2), 1)$ and $f \in (\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ be restricted at infinity. Then $f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ if and only if there exists a positive constant C such that for all $t_1, t_2 \in (0, \infty)$,*

$$(1.6) \quad \|e^{-t_1\sqrt{\Delta_\lambda}}e^{-t_2\sqrt{\Delta_\lambda}}(f)\|_{L^p(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda,1}(e^{-t_1\sqrt{\Delta_\lambda}}e^{-t_2\sqrt{\Delta_\lambda}}(f))\|_{L^p(\mathbb{R}_\lambda)} \\ + \|R_{\Delta_\lambda,2}(e^{-t_1\sqrt{\Delta_\lambda}}e^{-t_2\sqrt{\Delta_\lambda}}(f))\|_{L^p(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda,1}R_{\Delta_\lambda,2}(e^{-t_1\sqrt{\Delta_\lambda}}e^{-t_2\sqrt{\Delta_\lambda}}(f))\|_{L^p(\mathbb{R}_\lambda)} \leq C.$$

Based on the characterization of product Hardy space $H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)$ via Bessel Riesz transforms and the duality of $H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)$ with $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$, we directly have the fourth main result: the decomposition of $\text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$, whose proof is similar to the classical setting.

Theorem 1.10. *The following two statements are equivalent.*

- (i) $\varphi \in \text{BMO}_{\Delta_\lambda}(\mathbb{R}_\lambda)$;
- (ii) *There exist $g_i \in L^\infty(\mathbb{R}_\lambda)$, $i = 1, 2, 3, 4$, such that*

$$\varphi = g_1 + R_{\Delta_\lambda,1}(g_2) + R_{\Delta_\lambda,2}(g_3) + R_{\Delta_\lambda,1}R_{\Delta_\lambda,2}(g_4).$$

1.3. Structure and main methods of this paper. In Section 2, we first recall the known facts on product spaces of homogeneous type ([HLLu]) and then apply these to our setting \mathbb{R}_λ , including the Littlewood–Paley theory, Hardy and BMO spaces and atomic decompositions. We then provide the L^p -boundedness ($1 < p < \infty$) of the product Littlewood–Paley area functions Sf and $S_u f$ as defined in (1.3) and (1.5), respectively. In fact, we will prove this result for a more general Littlewood–Paley area functions and g -functions with the kernels of the operators inside satisfying certain size, smoothness and cancellation conditions which covers both the Bessel Poisson kernel and Bessel heat kernel. The main approach we use here is Calderón’s reproducing formula, almost orthogonality estimates and the Plancherel–Pólya type inequalities in the product setting.

In Section 3, we prove Theorem 1.2, our first main result. We note that the standard product Hardy spaces $H^p(\mathbb{R}_\lambda)$ (Definition 2.5) is a subset of a larger distribution space while our $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ is a completion of the initial subspace in $L^2(\mathbb{R}_\lambda)$. Hence, to show that $H^p(\mathbb{R}_\lambda)$ and $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ coincide, the main approach used here is via atomic decompositions. Following an idea in [CDLWY], we show that by choosing some particular function for Calderón’s reproducing formula (Proposition 3.1), we obtain the atomic decomposition for $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, which leads to $H^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda) = H_{\Delta_\lambda}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$ with equivalent norms (quasi-norms).

In Section 4, we present the proof of Theorem 1.6 by showing the following inequalities

$$(1.7) \quad \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)} \leq \|f\|_{H_{S_u}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\mathcal{N}_P}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{R_P}^p(\mathbb{R}_\lambda)} \\ \lesssim \|f\|_{H_{\mathcal{R}_h}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)},$$

i.e., all the norms above are equivalent.

Here the first inequality follows directly by definition. The fourth inequality follows from the well-known subordination formula which connects the Poisson kernel to the heat kernel. The fifth inequality follows directly from definition. The last inequality follows from the result that $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ coincides with the classical $H^p(\mathbb{R}_\lambda)$ (Theorem 1.2) and the fact that $H^p(\mathbb{R}_\lambda)$ has atomic decomposition. The main difficulties here are in the proofs of the second and third inequalities.

To prove the second inequality, we first point out that, to the best of our knowledge, the only one way up to now, to pass from the Littlewood–Paley area function to the non-tangential maximal function in the classical product setting is due to K. Merryfield [M]. The main technique in [M] relies on the construction of the function ψ in $C_c^\infty(\mathbb{R})$ according to any given $\phi \in C_c^\infty(\mathbb{R})$ with certain conditions, satisfying that for any $f \in L^2(\mathbb{R})$,

$$\partial_t(f * \phi_t(x)) = -\partial_x(f * \psi_t(x))$$

which is one of the Cauchy–Riemann equations in the classical setting. Here $\phi_t(x) := t^{-1}\phi(\frac{x}{t})$ and similar for ψ_t .

Based on the idea above, suppose $\phi \in C_c^\infty(\mathbb{R}_+)$ such that $\phi \geq 0$, $\text{supp}(\phi) \subset (0, 1)$, and $\int_0^\infty \phi(x) dm_\lambda(x) = 1$, we construct a function $\psi(t, x, y)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ by solving the following equation (which is one of the Cauchy–Riemann equations adapted to the Bessel setting):

$$\partial_t(\phi_t \sharp_\lambda f)(x) = \partial_x[\psi(f)(t, x)] + \frac{2\lambda}{x}\psi(f)(t, x),$$

where ϕ_t is the dilation of ϕ in the Bessel setting and $\psi(f)(t, x) := \int_{\mathbb{R}_+} \psi(t, x, y)f(y)dm_\lambda(y)$. Moreover, we show that $\psi(t, x, y)$ satisfies the required size, smoothness and cancellation conditions, and especially the support condition: $\text{supp} \psi(t, x, y) \subset \{t, x, y \in \mathbb{R}_+ : |x - y| < t\}$. Note that $\psi(f)$ here is no longer a Hankel convolution (for notation and details, we refer to Lemma 4.1).

To prove the third inequality, we borrow an idea from [YZ] in the one-parameter setting (see also [GLY1, GLY2]), to establish a product grand maximal function, which controls the non-tangential maximal function. Then, by using the reproducing formula and almost orthogonality estimates, we obtain that the L^p norm of the grand maximal function is bounded by that of the radial maximal function for $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$.

In Section 5, we prove our third main result, the Bessel Riesz transform characterizations of $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ (Theorems 1.8 and 1.9). The main ideas in the one-parameter setting are from Fefferman–Stein [FSt] (see also [St, Chapter III, Section 4.2] and [BDT]), where they obtained the result by studying the boundary value of the corresponding harmonic function and its conjugate. In our product setting, we consider the Bessel bi-harmonic function $u(t_1, t_2, x_1, x_2)$ and its three conjugate bi-harmonic functions v, w, z such that (u, v) and (w, z) satisfy the generalised Cauchy–Riemann equations (1.2) in the first group of variables (t_1, x_1) , and that (u, w) and (v, z) in the second group of variables (t_2, x_2) . Then, using Lemma 11 in [MSt], we obtain the harmonic majorants of the following four functions $\{u^2 + v^2\}^{\frac{p}{2}}$, $\{w^2 + z^2\}^{\frac{p}{2}}$, $\{u^2 + w^2\}^{\frac{p}{2}}$, $\{v^2 + z^2\}^{\frac{p}{2}}$ corresponding to the four groups of Cauchy–Riemann equations above respectively. By iteration, we obtain the harmonic majorant of the bi-harmonic function $\{u^2 + v^2 + w^2 + z^2\}^{\frac{p}{2}}$. Then, our main result follows from the properties of the Poisson semigroup $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$ and the standard approach ([St, Chapter III, Section 4.2]). To the best of our knowledge, the Hilbert transform characterizations not been addressed before for the classical Chang–Fefferman product Hardy space $H^p(\mathbb{R} \times \mathbb{R})$ when $p < 1$. We note that when $\lambda = 0$, our result and proof go back to $H^p(\mathbb{R} \times \mathbb{R})$ with minor modifications, and hence provide the characterizations of $H^p(\mathbb{R} \times \mathbb{R})$ via Hilbert transforms.

Throughout the whole paper, we denote by C and \tilde{C} positive constants which are independent of the main parameters, but they may vary from line to line. Constants with subscripts, such as C_0 and A_1 , do not change in different occurrences. For every $p \in (1, \infty)$, we denote by p' the conjugate of p , i.e., $\frac{1}{p'} + \frac{1}{p} = 1$. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we write $f \sim g$. For any $k \in \mathbb{R}_+$ and $I := I(x, r)$ for some $x, r \in (0, \infty)$, $kI := I(x, kr)$.

2. PRELIMINARIES

In this section, we first apply the known results on product spaces of homogeneous type developed in [HLLu, HLPW] to our setting on \mathbb{R}_λ . We then recall the properties of the Poisson kernels and conjugate Poisson kernels in the Bessel setting. Finally, we provide a continuous version of Littlewood–Paley theory, which will be used in the subsequent sections.

2.1. Product Hardy and BMO spaces on spaces of homogeneous type. To begin with, we point out that in [HLLu, HLPW], they considered the general setting of product spaces of homogeneous type $\mathbb{X} := (X_1, d_1, \mu_1) \times (X_2, d_2, \mu_2)$, and developed the test function spaces and distribution spaces, Calderón’s reproducing formula, Littlewood–Paley theory, product Hardy

and BMO spaces and atomic decompositions. For notational simplicity, we now apply all these results to our setting, i.e.,

$$\mathbb{X} := \mathbb{R}_\lambda := (\mathbb{R}_+, |\cdot|, dm_\lambda) \times (\mathbb{R}_+, |\cdot|, dm_\lambda).$$

We first observe that for any interval $I := I(x, r) \subset \mathbb{R}_+$, $m_\lambda(I) \sim x^{2\lambda}r + r^{2\lambda+1}$; moreover, from [DLMWY] we have that for any $I \subset \mathbb{R}_+$,

$$\min(2, 2^{2\lambda})m_\lambda(I) \leq m_\lambda(2I) \leq 2^{2\lambda+1}m_\lambda(I).$$

We now recall the definition of approximation to the identity.

Definition 2.1. We say that $\{S_k\}_{k \in \mathbb{Z}}$ is an approximation to the identity if $\lim_{k \rightarrow \infty} S_k = Id$, $\lim_{k \rightarrow -\infty} S_k = 0$ and moreover, the kernel $S_k(x, y)$ of S_k satisfies the following condition: for $\beta, \gamma \in (0, 1]$,

(A_i) for any $x, y, t \in \mathbb{R}_+$,

$$|S_k(x, y)| \lesssim \frac{1}{m_\lambda(I(x, 2^{-k})) + m_\lambda(I(y, 2^{-k})) + m_\lambda(I(x, |x - y|))} \left(\frac{2^{-k}}{|x - y| + 2^{-k}} \right)^\gamma;$$

(A_{ii}) for any $x, y, \tilde{y}, t \in \mathbb{R}_+$ with $|y - \tilde{y}| \leq (t + |x - y|)/2$,

$$|S_k(x, y) - S_k(x, \tilde{y})| + |S_k(y, x) - S_k(\tilde{y}, x)|$$

$$\lesssim \frac{1}{m_\lambda(I(x, 2^{-k})) + m_\lambda(I(y, 2^{-k})) + m_\lambda(I(x, |x - y|))} \left(\frac{|y - \tilde{y}|}{|x - y| + 2^{-k}} \right)^\beta \left(\frac{2^{-k}}{|x - y| + 2^{-k}} \right)^\gamma;$$

(A_{iii}) for any $x, y, \tilde{x}, \tilde{y}, t \in \mathbb{R}_+$ with $|y - \tilde{y}| \leq (t + |x - y|)/2$,

$$\begin{aligned} & |S_k(x, y) - S_k(x, \tilde{y}) - S_k(\tilde{x}, y) + S_k(\tilde{x}, \tilde{y})| \\ & \lesssim \frac{1}{m_\lambda(I(x, 2^{-k})) + m_\lambda(I(y, 2^{-k})) + m_\lambda(I(x, |x - y|))} \\ & \quad \times \left(\frac{|x - \tilde{x}|}{|x - y| + 2^{-k}} \right)^\beta \left(\frac{|y - \tilde{y}|}{|x - y| + 2^{-k}} \right)^\beta \left(\frac{2^{-k}}{|x - y| + 2^{-k}} \right)^\gamma; \end{aligned}$$

(A_{iv}) for any $t, x \in \mathbb{R}_+$,

$$\int_0^\infty S_k(x, y) dm_\lambda(y) = \int_0^\infty S_k(y, x) dm_\lambda(y) = 1.$$

One of the constructions of an approximation to the identity is due to Coifman, see [DJS]. We set $D_k := S_k - S_{k-1}$, and it is obvious that D_k satisfies (A_i), (A_{ii}) and (A_{iii}) and with

$$\int_0^\infty D_k(x, y) dm_\lambda(y) = \int_0^\infty D_k(y, x) dm_\lambda(y) = 0$$

for any $t, x \in \mathbb{R}_+$.

We now recall the results on Hardy spaces and Carleson measure spaces and related results developed in [HLLu]. We begin with the test function spaces and distribution spaces, and the one-parameter version of which was defined by Han, Müller, and Yang [HMY, HMY2], and then the product version by Han, Li, and Lu [HLLu].

Definition 2.2 ([HMY]). Consider the space $(\mathbb{R}_+, |\cdot|, dm_\lambda)$. Let $0 < \gamma, \beta \leq 1$ and $r > 0$. A function f defined on \mathbb{R}_+ is said to be a test function of type (x_0, r, β, γ) centered at $x_0 \in \mathbb{R}_+$ if f satisfies the following conditions:

- (i) $|f(x)| \leq C \frac{1}{V_r(x_0) + V(x, x_0)} \left(\frac{r}{r + d(x, x_0)} \right)^\gamma;$
- (ii) $|f(x) - f(y)| \leq C \left(\frac{d(x, y)}{r + d(x, x_0)} \right)^\beta \frac{1}{V_r(x_0) + V(x, x_0)} \left(\frac{r}{r + d(x, x_0)} \right)^\gamma$ for all $x, y \in \mathbb{R}_+$ with $d(x, y) \leq \frac{1}{2}(r + d(x, x_0))$.

Here $V(x, x_0) := m_\lambda(I(x, |x - x_0|))$. If f is a test function of type (x_0, r, β, γ) , we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ and the norm of $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ is defined by $\|f\|_{\mathcal{G}(x_0, r, \beta, \gamma)} := \inf\{C > 0 : \text{(i) and (ii) hold}\}$.

Now for any fixed $x_0 \in \mathbb{R}_+$, we denote $\mathcal{G}(\beta, \gamma) := \mathcal{G}(x_0, 1, \beta, \gamma)$ and by $\mathcal{G}_0(\beta, \gamma)$ the collection of all test functions in $\mathcal{G}(\beta, \gamma)$ with $\int_{\mathbb{R}_+} f(x) dm_\lambda(x) = 0$. Note that $\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$ with equivalent norms for all $x_1 \in \mathbb{R}_+$ and $r > 0$ and that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$.

Let $\mathring{\mathcal{G}}_1(\beta, \gamma)$ be the completion of the space $\mathcal{G}_0(1, 1)$ in the norm of $\mathcal{G}(\beta, \gamma)$ when $0 < \beta, \gamma < 1$. If $f \in \mathring{\mathcal{G}}_1(\beta, \gamma)$, we then define $\|f\|_{\mathring{\mathcal{G}}_1(\beta, \gamma)} := \|f\|_{\mathcal{G}(\beta, \gamma)}$. $(\mathring{\mathcal{G}}_1(\beta, \gamma))'$, the distribution space, is defined to be the set of all linear functionals L from $\mathring{\mathcal{G}}_1(\beta, \gamma)$ to \mathbb{C} with the property that there exists $C \geq 0$ such that for all $f \in \mathring{\mathcal{G}}_1(\beta, \gamma)$, $|L(f)| \leq C\|f\|_{\mathring{\mathcal{G}}_1(\beta, \gamma)}$.

Now we return to the product setting and recall the space of test functions and distributions on the product space \mathbb{R}_λ .

Definition 2.3 ([HLLu]). Let $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$, $0 < \gamma_1, \gamma_2, \beta_1, \beta_2 \leq 1$ and $r_1, r_2 > 0$. A function $f(x, y)$ defined on \mathbb{R}_λ is said to be a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ if for any fixed $y \in \mathbb{R}_+$, $f(x, y)$, as a function of the variable of x , is a test function in $\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)$ on \mathbb{R}_+ . Moreover, the following conditions are satisfied:

- (i) $\|f(\cdot, y)\|_{\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)} \leq \frac{C}{V_{r_2}(y_0) + V(y_0, y)} \left(\frac{r_2}{r_2 + d(y, y_0)} \right)^{\gamma_2}$;
- (ii) $\|f(\cdot, y) - f(\cdot, \tilde{y})\|_{\mathcal{G}(x_0, r_1, \beta_1, \gamma_1)} \leq \frac{C}{V_{r_2}(y_0) + V(y_0, y)} \left(\frac{d(y, \tilde{y})}{r_2 + d(y, y_0)} \right)^{\beta_2} \left(\frac{r_2}{r_2 + d(y, y_0)} \right)^{\gamma_2}$
for all $y, \tilde{y} \in \mathbb{R}_+$ with $d(y, \tilde{y}) \leq (r_2 + d(y, y_0))/2$.

Similarly, for any fixed $x \in \mathbb{R}_+$, $f(x, y)$, as a function of the variable of y , is a test function in $\mathcal{G}(y_0, r_2, \beta_2, \gamma_2)$ on \mathbb{R}_+ , and both properties (i) and (ii) also hold with x, y interchanged.

If f is a test function of type $(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$, we write $f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and the norm of f is defined by

$$\|f\|_{\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)} := \inf\{C : \text{(i), (ii) and (iii) hold}\}.$$

Similarly, we denote by $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class $\mathcal{G}(x_0, y_0; 1, 1; \beta_1, \beta_2; \gamma_1, \gamma_2)$ for any fixed $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$. Say that $f(x, y) \in \mathcal{G}_0(\beta_1, \beta_2; \gamma_1, \gamma_2)$ if

$$\int_{X_1} f(x, y) d\mu_1(x) = \int_{X_2} f(x, y) d\mu_2(y) = 0.$$

Note that $\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2) = \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with equivalent norms for all $(x_0, y_0) \in \mathbb{R}_+ \times \mathbb{R}_+$ and $r_1, r_2 > 0$ and that $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$.

Let $\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ be the completion of the space $\mathcal{G}_0(1, 1, 1, 1)$ in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $0 < \beta_i, \gamma_i < 1$. If $f \in \mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2)$, we then define $\|f\|_{\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2)} := \|f\|_{\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)}$.

We define the distribution space $(\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ by all linear functionals L from the space $\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to \mathbb{C} with the property that there exists $C \geq 0$ such that for all $f \in \mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2)$, $|L(f)| \leq C\|f\|_{\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2)}$.

Now we recall the Hardy space $H^p(\mathbb{R}_\lambda)$ in [HLLu] defined in terms of discrete Littlewood–Paley–Stein square function via a system of “dyadic cubes” in spaces of homogeneous type. We mention that in our current setting, we take the classical dyadic intervals as our dyadic system. That is, for each $k \in \mathbb{Z}$, $\mathcal{X}^k := \{\tau : I_\tau^k := (\tau 2^k, (\tau + 1)2^k]\}_{\tau \in \mathbb{Z}_+}$, where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

Definition 2.4 ([HLLu]). For $i = 1, 2$, let $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be approximations to the identity on \mathbb{R}_+ , and let $D_{k_i}^{(i)} := S_{k_i} - S_{k_i-1}$. For $f \in (\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ with $0 < \beta_i, \gamma_i < \theta_i$ for $i = 1, 2$, the discrete Littlewood–Paley–Stein square function $S_d(f)$ of f is defined by

$$S_d(f)(x, y) := \left\{ \sum_{k_1} \sum_{I_1 \in \mathcal{D}^{k_1+N_1}} \sum_{k_2} \sum_{I_2 \in \mathcal{D}^{k_2+N_2}} |D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(x_{I_1}, x_{I_2})|^2 \chi_{I_1}(x_1) \chi_{I_2}(x_2) \right\}^{\frac{1}{2}},$$

where x_{I_i} is the center of the dyadic interval I_i for $i = 1, 2$, and N_1 and N_2 are two large fixed positive numbers.

Definition 2.5 ([HLLu]). Suppose $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$. Suppose $0 < \beta_i, \gamma_i < 1$ for $i = 1, 2$. The Hardy space $H^p(\mathbb{R}_\lambda)$ is defined by

$$H^p(\mathbb{R}_\lambda) := \{f \in (\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))' : S_d(f) \in L^p(\mathbb{R}_\lambda)\},$$

with the norm (quasi-norm) $\|f\|_{H^p(\mathbb{R}_\lambda)} := \|S_d(f)\|_{L^p(\mathbb{R}_\lambda)}$.

Definition 2.6 ([HLLu, HLW]). Suppose $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$. Suppose $0 < \beta_i, \gamma_i < 1$ for $i = 1, 2$. The generalized Carleson measure space $\text{CMO}^p(\mathbb{R}_\lambda)$ is defined by the set of all $f \in (\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ such that

$$\|f\|_{\text{CMO}^p(\mathbb{R}_\lambda)} := \sup_{\Omega} \left\{ \frac{1}{\mu_\lambda(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1} \sum_{\alpha_1 \in \mathcal{D}^{k_1}} \sum_{k_2} \sum_{\alpha_2 \in \mathcal{D}^{k_2}} \left| D_{k_1}^{(1)} D_{k_2}^{(2)}(f)(x, y) \right|^2 \chi_{\{k_1, \alpha_1, k_2, \alpha_2: I_{\alpha_1}^{k_1} \times I_{\alpha_2}^{k_2} \subset \Omega\}}(k_1, \alpha_1, k_2, \alpha_2) \chi_{I_{\alpha_1}^{k_1} \times I_{\alpha_2}^{k_2}}(x, y) d\mu_\lambda(x, y) \right\}^{\frac{1}{2}} < \infty,$$

where Ω ranges over all open sets in $\mathbb{R}_+ \times \mathbb{R}_+$ with finite measure and χ_A is the characteristic function of a given set A .

Next we recall the atomic decomposition for $H^p(\mathbb{R}_\lambda)$ in [HLPW]. We call $R := I_{\tau_1}^{k_1} \times I_{\tau_2}^{k_2}$ a dyadic rectangle in $\mathbb{R}_+ \times \mathbb{R}_+$. Let $\Omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ be an open set of finite measure and $m_i(\Omega)$ denote the family of dyadic rectangles $R \subset \Omega$ which are maximal in the i th “direction”, $i = 1, 2$. Also we denote by $m(\Omega)$ the set of all maximal dyadic rectangles contained in Ω .

Definition 2.7 ([HLPW]). Suppose $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$. A function $a(x_1, x_2)$ defined on $\mathbb{R}_+ \times \mathbb{R}_+$ is called an atom of $H^p(\mathbb{R}_\lambda)$ if $a(x_1, x_2)$ satisfies:

- (1) $\text{supp } a \subset \Omega$, where Ω is an open set of $\mathbb{R}_+ \times \mathbb{R}_+$ with finite measure;
- (2) $\|a\|_{L^2(\mathbb{R}_\lambda)} \leq \mu_\lambda(\Omega)^{1/2-1/p}$;
- (3) a can be further decomposed into rectangular atoms a_R associated to dyadic rectangle $R := I_1 \times I_2$, satisfying the following
 - (i) there exist two constants \bar{C}_1 and \bar{C}_2 such that $\text{supp } a_R \subset \bar{C}_1 I_1 \times \bar{C}_2 I_2$;
 - (ii) $\int_{\mathbb{R}_+} a_R(x_1, x_2) dm_\lambda(x_1) = 0$ for a.e. $x_2 \in \mathbb{R}_+$ and $\int_{\mathbb{R}_+} a_R(x_1, x_2) dm_\lambda(x_2) = 0$ for a.e. $x_1 \in \mathbb{R}_+$;
 - (iii) $a = \sum_{R \in m(\Omega)} a_R$ and $\left(\sum_{R \in m(\Omega)} \|a_R\|_{L^2(\mathbb{R}_\lambda)}^2 \right)^{1/2} \leq \mu_\lambda(\Omega)^{1/2-1/p}$.

Theorem 2.8 ([HLPW]). Suppose $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$. Then $f \in L^2(\mathbb{R}_\lambda) \cap H^p(\mathbb{R}_\lambda)$ if and only if f has an atomic decomposition; that is,

$$f = \sum_{i=-\infty}^{\infty} \lambda_i a_i,$$

in the sense of both $H^p(\mathbb{R}_\lambda)$ and $L^2(\mathbb{R}_\lambda)$, where a_i are atoms and $\sum_i |\lambda_i|^p < \infty$. Moreover,

$$\|f\|_{H^p(\mathbb{R}_\lambda)} \approx \inf \left\{ \sum_{i=-\infty}^{\infty} |\lambda_i|^p \right\}^{\frac{1}{p}},$$

where the infimum is taken over all decompositions as above and the implicit constants are independent of the $L^2(\mathbb{R}_\lambda)$ and $H^p(\mathbb{R}_\lambda)$ norms of f .

2.2. Poisson kernel and conjugate Poisson kernel in the Bessel setting $(\mathbb{R}_+, |\cdot|, dm_\lambda)$.

Recall that $P_t^{[\lambda]}(f) := e^{-t\sqrt{\Delta_\lambda}} f = P_t^{[\lambda]} \#_\lambda f$ and $W_t^{[\lambda]}(f) := e^{-t\Delta_\lambda} f = W_{\sqrt{2t}}^{[\lambda]} \#_\lambda f$, where

$$P^{[\lambda]}(x) := \frac{2\lambda\Gamma(\lambda)}{\Gamma(\lambda+1/2)\sqrt{\pi}} \frac{1}{(1+x^2)^{\lambda+1}}$$

and $W^{[\lambda]}(x) := 2^{(1-2\lambda)/2} \exp(-x^2/2) / \Gamma(\lambda+1/2)$ and $f \in L^1(\mathbb{R}_+, dm_\lambda)$. For f and $\varphi \in L^1(\mathbb{R}_+, dm_\lambda)$, their Hankel convolution is defined by setting, for all $x, t \in (0, \infty)$,

$$(2.1) \quad \Phi_{t,\lambda} f(x) := \varphi_t \#_\lambda f(x) := \int_0^\infty f(y) \tau_x^{[\lambda]} \varphi_t(y) dm_\lambda(y),$$

where for $t, x \in (0, \infty)$, $\varphi_t(y) := t^{-2\lambda-1} \varphi(y/t)$ and $\tau_x^{[\lambda]} \varphi_t(y)$ denotes the Hankel translation of $\varphi_t(y)$, that is,

$$(2.2) \quad \tau_x^{[\lambda]} \varphi_t(y) := c_\lambda \int_0^\pi \varphi_t \left(\sqrt{x^2 + y^2 - 2xy \cos \theta} \right) (\sin \theta)^{2\lambda-1} d\theta$$

with $c_\lambda := \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda)\sqrt{\pi}}$, see [BDT, pp. 200-201] or [H].

Moreover, we recall that $\{e^{-t\Delta_\lambda}\}_{t>0}$ or $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$ have the following properties; see [BDT, YY, WYZ].

Lemma 2.9. *Let $\{T_t\}_{t>0}$ be one of $\{e^{-t\Delta_\lambda}\}_{t>0}$ or $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$. Then $\{T_t\}_{t>0}$ is a symmetric diffusion semigroup satisfying that $T_t T_s = T_s T_t$ for any $t, s \in (0, \infty)$, $T_0 = Id$, the identity operator, $\lim_{t \rightarrow 0} T_t f = f$ in $L^2(\mathbb{R}_+, dm_\lambda)$ and*

- (Si) $\|T_t f\|_{L^p(\mathbb{R}_+, dm_\lambda)} \leq \|f\|_{L^p(\mathbb{R}_+, dm_\lambda)}$ for all $p \in [1, \infty]$ and $t \in (0, \infty)$;
- (Sii) $T_t f \geq 0$ for all $f \geq 0$ and $t \in (0, \infty)$;
- (Siii) $T_t(1) = 1$ for all $t \in (0, \infty)$.

Next we recall the definitions of the Poisson kernel and conjugate Poisson kernel. For any $t, x, y \in (0, \infty)$,

$$P_t^{[\lambda]} f(x) := \int_0^\infty P_t^{[\lambda]}(x, y) f(y) y^{2\lambda} dy,$$

where

$$P_t^{[\lambda]}(x, y) = \frac{2\lambda t}{\pi} \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} d\theta.$$

See [W, BDT].

If $f \in L^p(\mathbb{R}_+, dm_\lambda)$, $p \in [1, \infty)$, the Δ_λ -conjugate of f is defined by setting, for any $t, x, y \in (0, \infty)$,

$$(2.3) \quad Q_t^{[\lambda]}(f)(x) := \int_0^\infty Q_t^{[\lambda]}(x, y) f(y) dm_\lambda(y),$$

where

$$Q_t^{[\lambda]}(x, y) := -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 + t^2 - 2xy \cos \theta)^{\lambda+1}} d\theta;$$

see [MSt, p. 84]. We point out that there exists the boundary value function $\lim_{t \rightarrow 0} Q_t^{[\lambda]}(f)(x)$ for almost every $x \in (0, \infty)$ (see [MSt, p. 84]), which is defined to be the Riesz transform $R_{\Delta_\lambda}(f)$, i.e.,

$$(2.4) \quad R_{\Delta_\lambda}(f)(x) := \lim_{t \rightarrow 0} Q_t^{[\lambda]}(f)(x) = \int_{\mathbb{R}_+} -\frac{2\lambda}{\pi} \int_0^\pi \frac{(x - y \cos \theta)(\sin \theta)^{2\lambda-1}}{(x^2 + y^2 - 2xy \cos \theta)^{\lambda+1}} d\theta f(y) dm_\lambda(y).$$

Moreover, we note that $u(t, x) := P_t^{[\lambda]}(f)(x)$ satisfies (1.1) and that $u(t, x) := P_t^{[\lambda]}(f)(x)$ and $v(t, x) := Q_t^{[\lambda]}(f)(x)$ satisfy the Cauchy–Riemann equations (1.2).

Proposition 2.10 ([YY, WYZ]). *For any fixed t and $x \in \mathbb{R}_+$, $P_t^{[\lambda]}(x, \cdot)$, $Q_t^{[\lambda]}(x, \cdot)$, $t\partial_t P_t^{[\lambda]}(x, \cdot)$ and $t\partial_y P_t^{[\lambda]}(x, \cdot)$ as functions of x are in $\mathring{\mathcal{G}}_1(\beta, \gamma)$ for all $\beta, \gamma \in (0, 1]$; symmetrically, for any fixed t and $y \in \mathbb{R}_+$, $t\partial_t P_t^{[\lambda]}(\cdot, y)$ and $t\partial_y P_t^{[\lambda]}(\cdot, y)$ are in $\mathring{\mathcal{G}}_1(\beta, \gamma)$ for all $\beta, \gamma \in (0, 1]$.*

Based on Definition 2.3 and Proposition 2.10, we further point out that

Proposition 2.11. *For any fixed t_1, t_2, x_1 and $x_2 \in \mathbb{R}_+$,*

$$P_{t_1}^{[\lambda]}(x_1, \cdot)P_{t_2}^{[\lambda]}(x_2, \cdot), P_{t_1}^{[\lambda]}(x_1, \cdot)Q_{t_2}^{[\lambda]}(x_2, \cdot), Q_{t_1}^{[\lambda]}(x_1, \cdot)P_{t_2}^{[\lambda]}(x_2, \cdot), Q_{t_1}^{[\lambda]}(x_1, \cdot)Q_{t_2}^{[\lambda]}(x_2, \cdot)$$

as functions of (x_1, x_2) is in $\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ for all $\beta_1, \gamma_1, \beta_2, \gamma_2 \in (0, 1]$.

2.3. L^p boundedness of the product Littlewood–Paley area functions and square functions. In this subsection, we provide the L^p boundedness of a general version of the product Littlewood–Paley area functions and square functions for $1 < p < \infty$, which covers $S(f)$ and $S_u(f)$ defined in (1.3) and (1.5), respectively.

To begin with, for $i = 1, 2$, we consider the integral operators $\{Q_{t_i}^{(i)}\}_{t_i > 0}$ associated with the kernels $K_{t_i}^{(i)}(x_i, y_i)$. Assume that $K_{t_i}^{(i)}(x_i, y_i)$ satisfies the following properties (for the sake of simplicity, when we state these properties we drop the superscript i):

(K_i) for any $x, y, t \in \mathbb{R}_+$,

$$|K_t(x, y)| \lesssim \frac{1}{m_\lambda(I(x, t)) + m_\lambda(I(y, t)) + m_\lambda(I(x, |x - y|))} \frac{t}{|x - y| + t};$$

(K_{ii}) for any $x, y, \tilde{y}, t \in \mathbb{R}_+$ with $|y - \tilde{y}| \leq (t + |x - y|)/2$,

$$|K_t(x, y) - K_t(x, \tilde{y})| \lesssim \frac{1}{m_\lambda(I(x, t)) + m_\lambda(I(y, t)) + m_\lambda(I(x, |x - y|))} \frac{t|y - \tilde{y}|}{(|x - y| + t)^2};$$

(K_{iii}) for any $t, x \in \mathbb{R}_+$,

$$K_t(1)(x) := \int_0^\infty K_t(x, y) dm_\lambda(y) = 0.$$

We now provide the L^p (for $1 < p < \infty$) boundedness of the product Littlewood–Paley square functions associated with the operators $Q_{t_1}^{(1)}$ and $Q_{t_2}^{(2)}$, which will be needed in Section 4. To be precise, we have

Theorem 2.12. *Let $p \in (1, \infty)$, $Q_{t_1}^{(1)}$ and $Q_{t_2}^{(2)}$ be the same as above. Then for every $g \in L^p(\mathbb{R}_\lambda)$, and almost all x_2 , we have that*

$$(2.5) \quad \left\| \left\{ \int_0^\infty \left| Q_{t_1}^{(1)}(g(\cdot, x_2))(x_1) \right|^2 \frac{dt_1}{t_1} \right\}^{1/2} \right\|_{L^p(\mathbb{R}_+, dm_\lambda(x_1))} \lesssim \|g(x_1, x_2)\|_{L^p(\mathbb{R}_+, dm_\lambda(x_1))},$$

and similar result holds for $Q_{t_2}^{(2)}$, and that

$$(2.6) \quad \left\| \left\{ \int_0^\infty \int_0^\infty \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(\cdot, \cdot) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|g\|_{L^p(\mathbb{R}_\lambda)}.$$

Proof. To begin with, we need to apply the general product discrete Littlewood–Paley theory on spaces of homogeneous type to our Bessel setting, by considering $(X_i, d_i, \mu_i) := (\mathbb{R}_+, |\cdot|, dm_\lambda)$ for $i = 1, 2$, i.e., $\mathbb{X} := \mathbb{R}_\lambda$. We note that in this product setting \mathbb{X} , we already have the discrete Littlewood–Paley theory (we refer to Theorem 2.14 and Proposition 2.16 in [HLLu]), stated as follows: for $1 < p < \infty$,

$$(2.7) \quad \|S_d(g)\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|g\|_{L^p(\mathbb{R}_\lambda)}.$$

We first prove (2.6). To this end, it suffices to prove that

$$(2.8) \quad \left\| \left\{ \int_0^\infty \int_0^\infty \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(\cdot, \cdot) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|S_d(g)\|_{L^p(\mathbb{R}_\lambda)}.$$

To see this, we now recall Calderón’s reproducing formula (Theorem 2.9 in [HLLu])

$$(2.9) \quad g(x_1, x_2) = \sum_{k_1} \sum_{I_1 \in \mathcal{X}^{k_1+N_1}} \sum_{k_2} \sum_{I_2 \in \mathcal{X}^{k_2+N_2}} m_\lambda(I_1) m_\lambda(I_2) \\ \times \tilde{D}_{k_1}^{(1)}(x_1, x_{I_1}) \tilde{D}_{k_2}^{(2)}(x_2, x_{I_2}) D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(x_{I_1}, x_{I_2}),$$

where the series converges in the sense of $L^p(\mathbb{R}_\lambda)$, for $i := 1, 2$, $\tilde{D}_{k_i}^{(i)}$ satisfies the same size, smoothness and cancellation conditions as $D_{k_i}^{(i)}$ does.

Observe that $Q_{t_1}^{(1)} Q_{t_2}^{(2)}$ is bounded on $L^p(\mathbb{R}_\lambda)$. Applying Calderón’s reproducing formula to the left-hand side of (2.8), we obtain that

$$(2.10) \quad \mathbb{L} := \int_0^\infty \int_0^\infty \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(x_1, x_2) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \\ = \sum_{\tilde{k}_1 \in \mathbb{Z}} \sum_{\tilde{k}_2 \in \mathbb{Z}} \int_{2^{-\tilde{k}_1-1}}^{2^{-\tilde{k}_1}} \int_{2^{-\tilde{k}_2-1}}^{2^{-\tilde{k}_2}} \left| \sum_{k_1} \sum_{I_1 \in \mathcal{X}^{k_1+N_1}} \sum_{k_2} \sum_{I_2 \in \mathcal{X}^{k_2+N_2}} m_\lambda(I_1) m_\lambda(I_2) \right. \\ \left. \times Q_{t_1}^{(1)}(\tilde{D}_{k_1}^{(1)}(\cdot, x_{I_1}))(x_1) Q_{t_2}^{(2)}(\tilde{D}_{k_2}^{(2)}(\cdot, x_{I_2}))(x_2) D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(x_{I_1}, x_{I_2}) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2}.$$

Note that in this case, $t_1 \sim 2^{-\tilde{k}_1}$ and $t_2 \sim 2^{-\tilde{k}_2}$, hence,

$$Q_{t_1}^{(1)}(\tilde{D}_{k_1}^{(1)}(\cdot, x_{I_1}))(x_1) Q_{t_2}^{(2)}(\tilde{D}_{k_2}^{(2)}(\cdot, x_{I_2}))(x_2)$$

satisfies the following almost orthogonality estimate (see Lemma 2.11 in [HLLu]): for $\epsilon \in (0, 1)$,

$$(2.11) \quad \left| Q_{t_1}^{(1)}(\tilde{D}_{k_1}^{(1)}(\cdot, x_{I_1}))(x_1) Q_{t_2}^{(2)}(\tilde{D}_{k_2}^{(2)}(\cdot, x_{I_2}))(x_2) \right| \\ \lesssim \prod_{i=1}^2 2^{-|k_i - \tilde{k}_i|\epsilon} \left(\frac{2^{-k_i} + 2^{-\tilde{k}_i}}{|x_i - x_{I_i}| + 2^{-k_i} + 2^{-\tilde{k}_i}} \right)^\epsilon \\ \times \frac{1}{m_\lambda(I(x_i, 2^{-k_i} + 2^{-\tilde{k}_i})) + m_\lambda(I(x_{I_i}, 2^{-k_i} + 2^{-\tilde{k}_i})) + m_\lambda(I(x_i, |x_i - x_{I_i}|))}.$$

Note that the right-hand side of the above inequality is independent of t_1 and t_2 . By substituting (2.11) back to the right-hand side of (2.10), we have that

$$\mathbb{L} \lesssim \sum_{\tilde{k}_1} \sum_{\tilde{k}_2} \left| \sum_{k_1} \sum_{I_1 \in \mathcal{X}^{k_1+N_1}} \sum_{k_2} \sum_{I_2 \in \mathcal{X}^{k_2+N_2}} \prod_{i=1}^2 m_\lambda(I_i) 2^{-|k_i - \tilde{k}_i|\epsilon} \left(\frac{2^{-k_i} + 2^{-\tilde{k}_i}}{|x_i - x_{I_i}| + 2^{-k_i} + 2^{-\tilde{k}_i}} \right)^\epsilon \right. \\ \left. \times \frac{1}{m_\lambda(I(x_i, 2^{-k_i} + 2^{-\tilde{k}_i})) + m_\lambda(I(x_{I_i}, 2^{-k_i} + 2^{-\tilde{k}_i})) + m_\lambda(I(x_i, |x_i - x_{I_i}|))} \right| \\ \times \left| D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(x_{I_1}, x_{I_2}) \right|^2.$$

Then, based on the estimates in the proof of Theorem 2.10 in [HLLu, pp. 335–336], we have the following estimate:

$$\begin{aligned} \mathbb{L} &\lesssim \sum_{k_1} \sum_{k_2} 2^{-|k_1 - \tilde{k}_1| \epsilon} 2^{-|k_2 - \tilde{k}_2| \epsilon} 2^{[(k_1 \wedge \tilde{k}_1) - k_1](2\lambda+1)(1-\frac{1}{r})} 2^{[(k_2 \wedge \tilde{k}_2) - k_2](2\lambda+1)(1-\frac{1}{r})} \\ &\quad \times \left[\mathcal{M}_1 \left(\sum_{I_1 \in \mathcal{J}^{k_1+N_1}} \mathcal{M}_2 \left(\sum_{I_2 \in \mathcal{J}^{k_2+N_2}} \inf_{\substack{y_1 \in I_1 \\ y_2 \in I_2}} \left| D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(y_1, y_2) \right|^r \chi_{I_2}(\cdot) \right) (x_2) \chi_{I_1}(\cdot) \right) (x_1) \right]^{\frac{2}{r}}, \end{aligned}$$

where $r < 1$, $a \wedge b := \min\{a, b\}$, and

$$\mathcal{M}_1 f(x_1, x_2) := \sup_{I \ni x_1} \frac{1}{m_\lambda(I)} \int_I |f(y_1, x_2)| dm_\lambda(y_1),$$

and

$$\mathcal{M}_2 f(x_1, x_2) := \sup_{J \ni x_2} \frac{1}{m_\lambda(J)} \int_J |f(x_1, y_2)| dm_\lambda(y_2).$$

By taking the square root and then the L^p norm on both sides of the above inequality and then using Fefferman–Stein’s vector-valued maximal function inequality ([HLLu]), we obtain that

$$\begin{aligned} (2.12) \quad &\left\| \left\{ \int_0^\infty \int_0^\infty \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(\cdot, \cdot) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}^{1/2} \right\|_{L^p(\mathbb{R}_\lambda)} \\ &\lesssim \left\| \left\{ \sum_{k_1} \sum_{I_1 \in \mathcal{J}^{k_1+N_1}} \sum_{k_2} \sum_{I_2 \in \mathcal{J}^{k_2+N_2}} \left| D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(x_{I_1}, x_{I_2}) \right|^2 \chi_{I_1}(\cdot) \chi_{I_2}(\cdot) \right\}^{1/2} \right\|_{L^p(\mathbb{R}_\lambda)} \\ &= \|S_d(g)\|_{L^p(\mathbb{R}_\lambda)}, \end{aligned}$$

which implies that (2.8) holds. Hence, we have that (2.6) holds.

Following the same steps above, we now sketch the proof of (2.5). Applying the following version of Calderón’s reproducing formula

$$(2.13) \quad g(x_1, x_2) = \sum_{k_1} \sum_{I_1 \in \mathcal{J}^{k_1+N_1}} m_\lambda(I_1) \tilde{D}_{k_1}^{(1)}(x_1, x_{I_1}) D_{k_1}^{(1)}(g)(x_{I_1}, x_2)$$

to the left-hand side of (2.5), we obtain that

$$\begin{aligned} (2.14) \quad \tilde{\mathbb{L}} &:= \int_0^\infty \left| Q_{t_1}^{(1)}(g)(x_1, x_2) \right|^2 \frac{dt_1}{t_1} \\ &= \sum_{\tilde{k}_1 \in \mathbb{Z}} \sum_{\tilde{k}_2 \in \mathbb{Z}} \int_{2^{-\tilde{k}_1-1}}^{2^{-\tilde{k}_1}} \left| \sum_{k_1} \sum_{I_1 \in \mathcal{J}^{k_1+N_1}} m_\lambda(I_1) Q_{t_1}^{(1)}(\tilde{D}_{k_1}^{(1)}(\cdot, x_{I_1}))(x_1) D_{k_1}^{(1)}(g)(x_{I_1}, x_2) \right|^2 \frac{dt_1}{t_1}. \end{aligned}$$

Then using the almost orthogonality estimate for $Q_{t_1}^{(1)}(\tilde{D}_{k_1}^{(1)}(x_1, x_{I_1}))$, and based on the estimates in the proof of Theorem 2.10 in [HLLu, pp. 335–336], we have the following estimate:

$$\tilde{\mathbb{L}} \lesssim \sum_{k_1} 2^{-|k_1 - \tilde{k}_1| \epsilon} 2^{[(k_1 \wedge \tilde{k}_1) - k_1](2\lambda+1)(1-\frac{1}{r})} \left[\mathcal{M}_1 \left(\sum_{I_1 \in \mathcal{J}^{k_1+N_1}} \inf_{y_1 \in I_1} \left| D_{k_1}^{(1)}(g)(y_1, x_2) \right|^r \chi_{I_1}(\cdot) \right) (x_1) \right]^{\frac{2}{r}},$$

where $r < 1$ and the Fefferman–Stein vector-valued maximal function inequality, we obtain that (2.5) holds. Similarly, we can obtain that (2.5) holds for $Q_{t_2}^{(2)}$. \square

Based on the proof of Theorem 2.12 above, we have the following estimates related to the Littlewood–Paley g -function.

Proposition 2.13. *Let $p \in \left(\frac{2\lambda+1}{2\lambda+2}, 1\right]$, $Q_{t_1}^{(1)}$ and $Q_{t_2}^{(2)}$ be the same as above. Then for every $g \in H^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$, we have that*

$$(2.15) \quad \left\| \left\{ \int_0^\infty \int_0^\infty \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(\cdot, \cdot) \right|^2 \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|S_d(g)\|_{L^p(\mathbb{R}_\lambda)}.$$

Proof. Following the same proof of Theorem 2.12 above, and noting that based on the estimates in the proof of Theorem 2.10 in [HLLu, pp. 335–336], we have the following estimate:

$$\begin{aligned} \mathbb{L} &\lesssim \sum_{k_1} \sum_{k_2} 2^{-|k_1 - \tilde{k}_1| \epsilon} 2^{-|k_2 - \tilde{k}_2| \epsilon} 2^{[(k_1 \wedge \tilde{k}_1) - k_1](2\lambda+1)(1-\frac{1}{r})} 2^{[(k_2 \wedge \tilde{k}_2) - k_2](2\lambda+1)(1-\frac{1}{r})} \\ &\quad \times \left[\mathcal{M}_1 \left(\sum_{I_1 \in \mathcal{I}^{k_1+N_1}} \mathcal{M}_2 \left(\sum_{I_2 \in \mathcal{I}^{k_2+N_2}} \inf_{\substack{y_1 \in I_1 \\ y_2 \in I_2}} \left| D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(y_1, y_2) \right|^r \chi_{I_2}(\cdot) \right) (x_2) \chi_{I_1}(\cdot) \right) (x_1) \right]^{\frac{2}{r}} \end{aligned}$$

for $\frac{2\lambda+1}{2\lambda+2} < r < p$, where \mathbb{L} is defined as in (2.10). Thus, we obtain that (2.12) holds for $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$, which implies (2.15). \square

Next we provide the L^p (for $1 < p < \infty$) boundedness of the product Littlewood–Paley area functions associated with the operators $Q_{t_1}^{(1)}$ and $Q_{t_2}^{(2)}$.

Theorem 2.14. *Let $Q_{t_1}^{(1)}$ and $Q_{t_2}^{(2)}$ be the same as above. Then for $1 < p < \infty$ and for every $g \in L^p(\mathbb{R}_\lambda)$, we have that*

$$(2.16) \quad \left\| \left\{ \iint_{\Gamma(x)} \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(y_1, y_2) \right|^2 \frac{d\mu_\lambda(y_1, y_2) dt_1 dt_2}{t_1 m_\lambda(I(x_1, t_1)) t_2 m_\lambda(I(x_2, t_2))} \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|g\|_{L^p(\mathbb{R}_\lambda)},$$

where $\Gamma(x) := \Gamma(x_1) \times \Gamma(x_2)$.

Proof. The proof of this theorem is similar to that of Theorem 2.12 and so we briefly sketch the proof. From Calderón’s reproducing formula (2.9) and the almost orthogonality estimates (2.11), we have

$$\begin{aligned} &\iint_{\Gamma(x)} \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(y_1, y_2) \right|^2 \frac{d\mu_\lambda(y_1, y_2) dt_1 dt_2}{t_1 m_\lambda(I(x_1, t_1)) t_2 m_\lambda(I(x_2, t_2))} \\ &\lesssim \sum_{k_1, k_2} \left[\mathcal{M}_1 \left(\sum_{I_1 \in \mathcal{I}^{k_1+N_1}} \mathcal{M}_2 \left(\sum_{I_2 \in \mathcal{I}^{k_2+N_2}} \inf_{\substack{y_1 \in I_1 \\ y_2 \in I_2}} \left| D_{k_1}^{(1)} D_{k_2}^{(2)}(g)(y_1, y_2) \right|^r \chi_{I_2}(\cdot) \right) (x_2) \chi_{I_1}(\cdot) \right) (x_1) \right]^{\frac{2}{r}} \end{aligned}$$

which implies that the left-hand side of (2.16) is bounded by $\|S_d(g)\|_{L^p(\mathbb{R}_\lambda)}$, which together with (2.7), finishes the proof of Theorem 2.14. \square

Note that from Proposition 2.10 and Lemma 2.9 (S_{iii}), we see that the kernels of $S(f)$ and $S_u(f)$ satisfy the conditions (K_i) , (K_{ii}) and (K_{iii}) listed above. As a direct consequence of Theorem 2.14, we have

Theorem 2.15. *The product Littlewood–Paley area functions $S(f)$ and $S_u(f)$ are bounded operators on $L^p(\mathbb{R}_\lambda)$, $1 < p < \infty$.*

3. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2. The main approach here is to show that $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ has a particular atomic decomposition as in Theorem 2.8

To begin with, we recall the following construction of ψ in [DY1]. Let $\varphi := -\pi i \chi_{\frac{1}{2} < |x| < 1}$ and ψ the Fourier transform of φ . That is,

$$\psi(s) := s^{-1}(2 \sin(s/2) - \sin s).$$

Consider the operator

$$(3.1) \quad \psi(t\sqrt{\Delta_\lambda}) := (t\sqrt{\Delta_\lambda})^{-1} \left[2 \sin(t\sqrt{\Delta_\lambda}/2) - \sin(t\sqrt{\Delta_\lambda}) \right].$$

Proposition 3.1 ([DY1]). *For all $t \in (0, \infty)$, $\psi(t\sqrt{\Delta_\lambda})(1) = 0$ and the kernel $K_{\psi(t\sqrt{\Delta_\lambda})}$ of $\psi(t\sqrt{\Delta_\lambda})$ has support contained in $\{(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : |x - y| \leq t\}$.*

We now turn to the proof of our first main result, Theorem 1.2.

Proof of Theorem 1.2. Note that from the definition of $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, we have $L^2(\mathbb{R}_\lambda) \cap H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ is dense in $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$. Moreover, from Proposition 2.19 in [HLLu], we have that $L^2(\mathbb{R}_\lambda) \cap H^p(\mathbb{R}_\lambda)$ is dense in $H^p(\mathbb{R}_\lambda)$. Thus, by a density argument, it suffices to show that

$$(3.2) \quad L^2(\mathbb{R}_\lambda) \cap H_{\Delta_\lambda}^p(\mathbb{R}_\lambda) = L^2(\mathbb{R}_\lambda) \cap H^p(\mathbb{R}_\lambda)$$

with equivalent norms.

We first prove that for every $f \in L^2(\mathbb{R}_\lambda) \cap H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, f belongs to $H^p(\mathbb{R}_\lambda)$ and

$$(3.3) \quad \|f\|_{H^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)}.$$

To prove this argument, it suffices to show that for every $f \in L^2(\mathbb{R}_\lambda) \cap H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, f has an atomic decomposition, with atoms satisfying the properties as in Definition 2.7.

To see this, we adapt the proof of the atomic decomposition as in Proposition 3.4 in [CDLWY] to our current setting of Bessel operators. We point out that in [CDLWY] they considered only the atomic decompositions for Hardy space $H_{L_1, L_2}^1(X_1 \times X_2)$ associated with operators L_1 and L_2 , but their methods also work for $p < 1$.

Let $f \in L^2(\mathbb{R}_\lambda) \cap H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$. For each $j \in \mathbb{Z}$, define

$$\Omega_j := \{(x_1, x_2) \in \mathbb{R}_\lambda : Sf(x_1, x_2) > 2^j\},$$

$$B_j := \{R := I_{\alpha_1}^{k_1} \times I_{\alpha_2}^{k_2} : \mu_\lambda(R \cap \Omega_j) > \mu_\lambda(R)/2, \mu_\lambda((R \cap \Omega_{j+1}) \leq \mu_\lambda(R)/2\},$$

and

$$\tilde{\Omega}_j := \{(x_1, x_2) \in \mathbb{R}_\lambda : \mathcal{M}_S(\chi_{\Omega-j})(x_1, x_2) > 1/2\}.$$

Here we use $I_{\alpha_i}^{k_i}$ ($i = 1, 2$) to denote the dyadic intervals as stated in Section 2 and \mathcal{M}_S is the maximal function defined by

$$(3.4) \quad \mathcal{M}_S(f)(x_1, x_2) := \sup_{\substack{I \ni x_1, J \ni x_2 \\ R := I \times J}} \frac{1}{\mu_\lambda(R)} \iint_R |f(y_1, y_2)| d\mu_\lambda(y_1, y_2),$$

where the supremum is taken over all rectangles $R := I \times J$ with intervals $I, J \subset \mathbb{R}_+$. For each dyadic rectangle $R := I_{\alpha_1}^{k_1} \times I_{\alpha_2}^{k_2}$ in \mathbb{R}_λ , the tent $T(R)$ is defined as

$$T(R) := \{(y_1, y_2, t_1, t_2) : (y_1, y_2) \in R, t_1 \in [2^{-k_1}, 2^{-k_1+1}], t_2 \in [2^{-k_2}, 2^{-k_2+1}]\}.$$

We now consider the following reproducing formula

$$f(x_1, x_2) = C_\psi \int_0^\infty \int_0^\infty \psi(t_1\sqrt{\Delta_\lambda})\psi(t_2\sqrt{\Delta_\lambda})(t_1\sqrt{\Delta_\lambda}e^{-t_1\sqrt{\Delta_\lambda}} \otimes t_2\sqrt{\Delta_\lambda}e^{-t_2\sqrt{\Delta_\lambda}})f(x_1, x_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2},$$

in the sense of $L^2(\mathbb{R}_\lambda)$, where $\psi(t_1\sqrt{\Delta_\lambda})$ and $\psi(t_2\sqrt{\Delta_\lambda})$ are defined as in (3.1) and C_ψ is a constant depending on ψ (see (3.13) in [CDLWY]). Then we have

$$(3.5) \quad f(x_1, x_2) = C_\psi \sum_{j \in \mathbb{Z}} \sum_{R \in B_j} \iiint_{T(R)} \psi(t_1\sqrt{\Delta_\lambda})(x_1, y_1) \psi(t_2\sqrt{\Delta_\lambda})(x_2, y_2)$$

$$\begin{aligned}
& (t_1 \sqrt{\Delta_\lambda} e^{-t_1 \sqrt{\Delta_\lambda}} \otimes t_2 \sqrt{\Delta_\lambda} e^{-t_2 \sqrt{\Delta_\lambda}})(f)(y_1, y_2) d\mu_\lambda(y_1, y_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2} \\
& =: \sum_{j \in \mathbb{Z}} \alpha_j a_j(x_1, x_2)
\end{aligned}$$

with

$$\begin{aligned}
\alpha_j &:= C_\psi \left\| \left(\sum_{R \in B_j} \int_0^\infty \int_0^\infty \left| (t_1 \sqrt{\Delta_\lambda} e^{-t_1 \sqrt{\Delta_\lambda}} \otimes t_2 \sqrt{\Delta_\lambda} e^{-t_2 \sqrt{\Delta_\lambda}}) f(\cdot, \cdot) \right|^2 \chi_{T(R)} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \right)^{\frac{1}{2}} \right\|_{L^2(\mathbb{R}_\lambda)} \\
& \quad \times \mu_\lambda(\tilde{\Omega}_j)^{1/p-1/2},
\end{aligned}$$

and

$$a_j(x_1, x_2) = \sum_{\bar{R} \in m(\tilde{\Omega}_j)} a_{j, \bar{R}}(x_1, x_2),$$

where

$$\begin{aligned}
a_{j, \bar{R}} &:= \sum_{R \in B_j, R \subset \bar{R}} \frac{1}{\alpha_j} \iiint_{T(R)} \psi(t_1 \sqrt{\Delta_\lambda})(x_1, y_1) \psi(t_2 \sqrt{\Delta_\lambda})(x_2, y_2) \\
& \quad (t_1 \sqrt{\Delta_\lambda} e^{-t_1 \sqrt{\Delta_\lambda}} \otimes t_2 \sqrt{\Delta_\lambda} e^{-t_2 \sqrt{\Delta_\lambda}})(f)(y_1, y_2) d\mu_\lambda(y_1, y_2) \frac{dt_1}{t_1} \frac{dt_2}{t_2}.
\end{aligned}$$

Following the proof of Proposition 3.4 in [CDLWY], we deduce that

$$(3.6) \quad \sum_j |\alpha_j|^p \leq C \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)}^p$$

and that for each j , a_j is supported in $\tilde{\Omega}_j$, and that $\|a_j\|_{L^2(\mathbb{R}_\lambda)} \leq \mu_\lambda(\tilde{\Omega}_j)^{1/2-1/p}$, which implies that a_j satisfies (1) and (2) in Definition 2.7. Moreover, each $a_{j, \bar{R}}(x_1, x_2)$ is supported in CR , where C is a fixed positive constant, and

$$\sum_{\bar{R} \in m(\tilde{\Omega}_j)} \|a_{j, \bar{R}}\|_{L^2(\mathbb{R}_\lambda)}^2 \leq \mu_\lambda(\tilde{\Omega}_j)^{1-2/p}.$$

By Proposition 3.1, we also see that

$$\int a_{j, \bar{R}}(x_1, x_2) dm_\lambda(x_1) = \int a_{j, \bar{R}}(x_1, x_2) dm_\lambda(x_2) = 0.$$

This shows that a_j satisfies (3) in Definition 2.7. Combining these results, we get that for each j , a_j is an atom as in Definition 2.7. Hence, applying the result of Theorem 2.8 and (3.5) and (3.6), we obtain that (3.3) holds.

Next we prove that for every $f \in L^2(\mathbb{R}_\lambda) \cap H^p(\mathbb{R}_\lambda)$, f belongs to $H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ and

$$(3.7) \quad \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H^p(\mathbb{R}_\lambda)}.$$

To see this, note that for every $f \in L^2(\mathbb{R}_\lambda) \cap H^p(\mathbb{R}_\lambda)$, from Theorem 2.8, we obtain that $f = \sum_k \lambda_k a_k$, where each a_k is an atom in Definition 2.7 and $\sum_k |\lambda_k|^p \lesssim \|f\|_{H^p(\mathbb{R}_\lambda)}^p$.

As pointed out in Section 2.3, $S(f)$ is bounded on $L^2(\mathbb{R}_\lambda)$ (Theorem 2.15), and the kernels of $S(f)$ satisfy the conditions (K_i), (K_{ii}) and (K_{iii}). Then, we have

$$\|S(a_k)\|_{L^p(\mathbb{R}_\lambda)} \lesssim 1,$$

where the implicit constant is independent of a_k . As a consequence, we have

$$\|S(f)\|_{L^p(\mathbb{R}_\lambda)}^p \leq \sum_k |\lambda_k|^p \|S(a_k)\|_{L^p(\mathbb{R}_\lambda)}^p \lesssim \|f\|_{H^p(\mathbb{R}_\lambda)}^p,$$

which implies that (3.7) holds.

Combining the results of (3.3) and (3.7), we get that (3.2) holds, with equivalent norms. This completes the proof of Theorem 1.2. \square

Based on the proof of Theorem 1.2 above, we now prove that the three versions of Hardy spaces, i.e., $H_{\Delta_\lambda,1}^p(\mathbb{R}_\lambda)$, $H_{\Delta_\lambda,2}^p(\mathbb{R}_\lambda)$ and $H_{\Delta_\lambda,3}^p(\mathbb{R}_\lambda)$, as define in Remark 1.3, coincide with $H^p(\mathbb{R}_\lambda)$.

Proposition 3.2. *For $p \in \left(\frac{2\lambda+1}{2\lambda+2}, 1\right]$, the Hardy spaces $H_{\Delta_\lambda,1}^p(\mathbb{R}_\lambda)$, $H_{\Delta_\lambda,2}^p(\mathbb{R}_\lambda)$ and $H_{\Delta_\lambda,3}^p(\mathbb{R}_\lambda)$ coincide with $H^p(\mathbb{R}_\lambda)$ and they have equivalent norms (or quasi-norms).*

Proof. We first consider $H_{\Delta_\lambda,1}^p(\mathbb{R}_\lambda)$ defined by the Littlewood–Paley area function via the heat semigroup $\{e^{-t\Delta_\lambda}\}_{t>0}$. Since the kernel of $\{e^{-t\Delta_\lambda}\}_{t>0}$ satisfies conditions (K_i), (K_{ii}) and (K_{iii}), following the proof of Theorem 1.2, we obtain that $H_{\Delta_\lambda,1}^p(\mathbb{R}_\lambda)$ coincides with $H^p(\mathbb{R}_\lambda)$ and they have equivalent norms (or quasi-norms).

Next we consider $H_{\Delta_\lambda,2}^p(\mathbb{R}_\lambda)$ defined by the Littlewood–Paley g -function via the Poisson semigroup $\{e^{-t\sqrt{\Delta_\lambda}}\}_{t>0}$. Suppose $f \in H^p(\mathbb{R}_\lambda)$. Then from Proposition 2.13, we obtain that $\|g(f)\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|S_d(f)\|_{L^p(\mathbb{R}_\lambda)}$, which shows that $H_{\Delta_\lambda,2}^p(\mathbb{R}_\lambda) \supseteq H^p(\mathbb{R}_\lambda)$. Conversely, suppose $f \in H_{\Delta_\lambda,2}^p(\mathbb{R}_\lambda)$, following the proof of Proposition 2.13, we obtain that $\|S_d(f)\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|g(f)\|_{L^p(\mathbb{R}_\lambda)}$ and hence $H_{\Delta_\lambda,2}^p(\mathbb{R}_\lambda) \subseteq H^p(\mathbb{R}_\lambda)$. As a consequence, we get that $H_{\Delta_\lambda,2}^p(\mathbb{R}_\lambda)$ coincides with $H^p(\mathbb{R}_\lambda)$ and they have equivalent norms. Similar argument holds for $H_{\Delta_\lambda,3}^p(\mathbb{R}_\lambda)$. \square

4. PROOF OF THEOREM 1.6

This section is devoted to the proof of Theorem 1.6. To this end, we will prove the chain of six inequalities as in (1.7) by the following six steps, respectively.

Step 1: $\|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)} \leq \|f\|_{H_{S_u}^p(\mathbb{R}_\lambda)}$ for $f \in H_{S_u}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$.

Note that from the definitions of the area functions Sf and $S_u f$ in (1.3) and (1.5) respectively, we have for $f \in L^2(\mathbb{R}_\lambda)$, $S(f)(x) \leq S_u(f)(x)$, which implies that $\|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)} \leq \|f\|_{H_{S_u}^p(\mathbb{R}_\lambda)}$.

Step 2: $\|f\|_{H_{S_u}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\mathcal{N}_P}^p(\mathbb{R}_\lambda)}$ for $f \in H_{\mathcal{N}_P}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$.

Recall again

$$\nabla_{t,x} u(t, x) := (\partial_t u, \partial_x u), \quad \Delta_{t,x} u(t, x) := \Delta_\lambda + \partial_t^2 = -\mathcal{D}^* \mathcal{D} + \partial_t^2,$$

where $\mathcal{D}^* := -\partial_x - \frac{2\lambda}{x} \partial_x$ is the formal adjoint operator of $\mathcal{D} := \partial_x$ in $L^2(\mathbb{R}_+, dm_\lambda)$.

Next we introduce the following lemma about finding the “conjugate pair” of functions (ϕ, ψ) , which plays a key role in this step.

Lemma 4.1. *Let $\phi \in C_c^\infty(\mathbb{R}_+)$ such that $\phi \geq 0$, $\text{supp}(\phi) \subset (0, 1)$ and*

$$\int_0^\infty \phi(x) dm_\lambda(x) = 1.$$

Then there exists a function $\psi(t, x, y)$ on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, such that

(i) for any function $f \in L^2(\mathbb{R}_+, dm_\lambda)$ and $t, x \in \mathbb{R}_+$,

$$(4.1) \quad \partial_t(\phi_t \sharp_\lambda f)(x) = -\mathcal{D}^*[\psi(f)(t, x)] = \partial_x[\psi(f)(t, x)] + \frac{2\lambda}{x} \psi(f)(t, x),$$

where

$$\psi(f)(t, x) := \int_0^\infty \psi(t, x, y) f(y) dm_\lambda(y);$$

(ii) for any t, x, y with $|x - y| \geq t$,

$$(4.2) \quad \psi(t, x, y) \equiv 0;$$

(iii) $\psi(t, x, y)$ satisfies the conditions (K_i), (K_{ii}) and (K_{iii}) as in Section 2.3.

Proof. First, by (2.2), we observe that

$$\begin{aligned} \partial_t \tau_x^{[\lambda]} \phi_t(y) &= -c_\lambda t^{-2\lambda-2} \int_0^\pi (\sin \theta)^{2\lambda-1} \left[(2\lambda+1) \phi \left(\frac{\sqrt{x^2+y^2-2xy \cos \theta}}{t} \right) \right. \\ &\quad \left. + \frac{\sqrt{x^2+y^2-2xy \cos \theta}}{t} \phi' \left(\frac{\sqrt{x^2+y^2-2xy \cos \theta}}{t} \right) \right] d\theta. \end{aligned}$$

Note that (4.1) holds if ψ satisfies that for any t, x, y ,

$$\partial_t \tau_x^{[\lambda]} \phi_t(y) = \partial_x \psi(t, x, y) + \frac{2\lambda}{x} \psi(t, x, y).$$

Thus, we define

$$\begin{aligned} (4.3) \quad \psi(t, x, y) &:= -c_\lambda t^{-2\lambda-2} x^{-2\lambda} \int_0^x \int_0^\pi (\sin \theta)^{2\lambda-1} \left[(2\lambda+1) \phi \left(\frac{\sqrt{w^2+y^2-2wy \cos \theta}}{t} \right) \right. \\ &\quad \left. + \frac{\sqrt{w^2+y^2-2wy \cos \theta}}{t} \phi' \left(\frac{\sqrt{w^2+y^2-2wy \cos \theta}}{t} \right) \right] d\theta w^{2\lambda} dw. \end{aligned}$$

Then it is easy to see that ψ satisfies the equation (4.1).

Now we prove that (4.2) holds. In fact, for all $x, y, z \in (0, \infty)$, let $\triangle(x, y, z)$ be the area of a triangle with sides x, y, z if such a triangle exists. And then we define

$$D(x, y, z) := c_\lambda 2^{2\lambda-2} (xyz)^{-2\lambda+1} [\triangle(x, y, z)]^{2\lambda-2}$$

if $\triangle(x, y, z) \neq 0$, and $D(x, y, z) := 0$ otherwise. By a change of variables argument, we obtain that

$$c_\lambda \int_0^\pi \phi_t \left(\sqrt{x^2+y^2-2xy \cos \theta} \right) (\sin \theta)^{2\lambda-1} d\theta = \int_0^\infty \phi_t(z) D(x, y, z) dm_\lambda(z).$$

Recall that for all $x, z \in (0, \infty)$,

$$(4.4) \quad \int_0^\infty D(x, y, z) dm_\lambda(y) = 1;$$

see in [H, p. 335, (6)]. By change of variables, we write

$$\begin{aligned} (4.5) \quad \psi(t, x, y) &= -t^{-2\lambda-2} x^{-2\lambda} \int_0^x \int_0^\infty D(w, y, z) \left[(2\lambda+1) \phi \left(\frac{z}{t} \right) + \frac{z}{t} \phi' \left(\frac{z}{t} \right) \right] z^{2\lambda} dz w^{2\lambda} dw \\ &= -t^{-1} x^{-2\lambda} \int_0^x \int_0^\infty D(w, y, z) \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] dz w^{2\lambda} dw \\ &= -t^{-1} x^{-2\lambda} \int_0^\infty \int_0^x D(w, y, z) \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] w^{2\lambda} dw dz. \end{aligned}$$

We first prove $\psi(t, x, y) = 0$ if $x > y + t$. To this end, recall that $\text{supp}(\phi) \subset (0, 1)$. Then $\phi(z/t) \neq 0$ only if $z \in (0, t)$. Also, by the definition of $D(w, y, z)$, we see that $D(w, y, z) \neq 0$ only if $|y - z| < w < y + z$. Then by (4.4) and the fact that $\phi \in C_c^\infty(\mathbb{R}_+)$, we have that

$$\begin{aligned} \psi(t, x, y) &= -t^{-1} x^{-2\lambda} \int_0^\infty \int_0^\infty D(w, y, z) \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] w^{2\lambda} dw dz \\ &= -t^{-1} x^{-2\lambda} \int_0^\infty \int_0^\infty D(w, y, z) w^{2\lambda} dw \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] dz \\ &= -t^{-1} x^{-2\lambda} \int_0^\infty \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] dz = 0. \end{aligned}$$

On the other hand, assume that $x < y - t$. Then by the compact support of ϕ , we see that $w \leq x < y - t \leq y - z \leq |y - z|$. This together with the definition of $D(w, y, z)$ implies that $\psi(t, x, y) = 0$.

Now we show ψ satisfies (K_i). By (4.2) and the doubling property of dm_λ , it suffices to show that for any t, x, y such that $|x - y| < t$,

$$(4.6) \quad |\psi(t, x, y)| \lesssim \frac{1}{m_\lambda(I(x, t))}.$$

From (4.5), (4.4) and $\phi \in C_c^\infty(\mathbb{R}_+)$, we deduce that

$$|\psi(t, x, y)| \lesssim t^{-1} x^{-2\lambda} \left| \int_0^\infty \int_0^x D(w, y, z) \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] w^{2\lambda} dw dz \right| \lesssim t^{-1} x^{-2\lambda}.$$

Moreover, if $x \leq t$, then from (4.5), (4.4) and $\phi \in C_c^\infty(\mathbb{R}_+)$, it follows that

$$\begin{aligned} |\psi(t, x, y)| &\lesssim t^{-2\lambda-2} x^{-2\lambda} \int_0^x \int_0^\infty D(w, y, z) \left| (2\lambda+1) \phi \left(\frac{z}{t} \right) + \frac{z}{t} \phi' \left(\frac{z}{t} \right) \right| z^{2\lambda} dz w^{2\lambda} dw \\ &\lesssim t^{-2\lambda-2} x^{-2\lambda} \int_0^x \int_0^\infty D(w, y, z) z^{2\lambda} dz w^{2\lambda} dw \lesssim t^{-2\lambda-2} x \lesssim t^{-2\lambda-1}. \end{aligned}$$

Thus, (K_i) holds.

Now we show ψ satisfies (K_{ii}). We first observe that by the assumption that $|y - \tilde{y}| \leq (t + |x - y|)/2$ and the doubling property of dm_λ , $|x - y| + t \sim |x - \tilde{y}| + t$ and

$$m_\lambda(I(x, t)) + m_\lambda(I(y, t)) + m_\lambda(I(x, |x - y|)) \sim m_\lambda(I(x, t)) + m_\lambda(I(\tilde{y}, t)) + m_\lambda(I(x, |x - \tilde{y}|)).$$

Based on these facts, we may further assume that $y > \tilde{y}$. For otherwise, it is sufficient to show that

$$(4.7) \quad |\psi(t, x, y) - \psi(t, x, \tilde{y})| \lesssim \frac{1}{m_\lambda(I(x, t)) + m_\lambda(I(\tilde{y}, t)) + m_\lambda(I(x, |x - \tilde{y}|))} \frac{t|y - \tilde{y}|}{(|x - \tilde{y}| + t)^2}.$$

Since $y > \tilde{y}$, if $x > y + t$, then $x > \tilde{y} + t$ and by (4.2), we see that $\psi(t, x, y) = 0$ and $\psi(t, x, \tilde{y}) = 0$. Similarly, if $x < \tilde{y} - t$, then $x < y - t$ and by (4.2) again, $\psi(t, x, y) = 0$ and $\psi(t, x, \tilde{y}) = 0$. Hence, (K_{ii}) holds trivially if $x > y + t$ or $x < \tilde{y} - t$. Moreover, observe that

$$[\tilde{y} - t, y + t] = [\tilde{y} - t, \tilde{y} + t] \cup [y - t, y + t].$$

Therefore, by similarity and (4.8), we only need to consider the case that $y - t \leq x \leq y + t$. It then suffices to show that

$$(4.8) \quad |\psi(t, x, y) - \psi(t, x, \tilde{y})| \lesssim \frac{1}{m_\lambda(I(y, t)) + m_\lambda(I(y, |x - y|))} \frac{|y - \tilde{y}|}{t}.$$

We write

$$\begin{aligned} &|\psi(t, x, y) - \psi(t, x, \tilde{y})| \\ &\lesssim t^{-2\lambda-2} x^{-2\lambda} \int_0^\pi \int_0^x (\sin \theta)^{2\lambda-1} \left| \phi \left(\frac{\sqrt{w^2 + y^2 - 2wy \cos \theta}}{t} \right) \right. \\ &\quad \left. - \phi \left(\frac{\sqrt{w^2 + \tilde{y}^2 - 2w\tilde{y} \cos \theta}}{t} \right) \right| w^{2\lambda} dw d\theta \\ &\quad + t^{-2\lambda-2} x^{-2\lambda} \int_0^\pi \int_0^x (\sin \theta)^{2\lambda-1} \left| \frac{\sqrt{w^2 + y^2 - 2wy \cos \theta}}{t} \phi' \left(\frac{\sqrt{w^2 + y^2 - 2wy \cos \theta}}{t} \right) \right. \\ &\quad \left. - \frac{\sqrt{w^2 + \tilde{y}^2 - 2w\tilde{y} \cos \theta}}{t} \phi' \left(\frac{\sqrt{w^2 + \tilde{y}^2 - 2w\tilde{y} \cos \theta}}{t} \right) \right| w^{2\lambda} dw d\theta \\ &=: \text{I} + \text{II}. \end{aligned}$$

We only consider the term I, since for the term II, we consider the function $\tilde{\phi}(x) := x\phi'(x)$, and then the form of II will be the same as I. We study the following four cases:

For the term I, we first note that from the mean value theorem,

$$\begin{aligned} \text{I} &\lesssim t^{-2\lambda-2} x^{-2\lambda} \int_0^\pi \int_0^x (\sin \theta)^{2\lambda-1} \left| \phi' \left(\frac{\sqrt{w^2 + \xi^2 - 2w\xi \cos \theta}}{t} \right) \right| \\ &\quad \times \frac{|\xi - w \cos \theta| |y - \tilde{y}|}{t \sqrt{w^2 + \xi^2 - 2w\xi \cos \theta}} w^{2\lambda} dw d\theta \\ &\lesssim t^{-2\lambda-2} x^{-2\lambda} \int_0^\pi \int_0^x (\sin \theta)^{2\lambda-1} \left| \phi' \left(\frac{\sqrt{w^2 + \xi^2 - 2w\xi \cos \theta}}{t} \right) \right| \frac{|y - \tilde{y}|}{t} w^{2\lambda} dw d\theta. \end{aligned}$$

To continue, we consider the following three cases.

Case (i) $t < y \leq 8t$. In this case, by $x < y + t$, we have $x < 2y \leq 16t$. Again, since $|\phi'(x)| \lesssim 1$, we get that

$$\begin{aligned} \text{I} &\lesssim x^{-2\lambda} |y - \tilde{y}| t^{-2\lambda-3} \int_0^\pi \int_0^x (\sin \theta)^{2\lambda-1} w^{2\lambda} dw d\theta \\ &\lesssim x^{-2\lambda} |y - \tilde{y}| t^{-2\lambda-3} x^{2\lambda+1} \\ &\lesssim |y - \tilde{y}| t^{-2\lambda-2} \\ &\lesssim \frac{1}{m_\lambda(I(y, t)) + m_\lambda(I(y, |x - y|))} \frac{|y - y'|}{t}. \end{aligned}$$

Case (ii) $8t < y$ and $|x - y| < y/2$. In this case,

$$|y - \xi| < |y - \tilde{y}| \leq \frac{t + |x - y|}{2} < \frac{5}{16}y,$$

which implies that $\xi \sim y \sim \tilde{y} \sim x$. Thus, we see that

$$\begin{aligned} \text{I} &\lesssim t^{-2\lambda-3} y^{-2\lambda} |y - \tilde{y}| \int_0^\infty \int_0^\infty D(\xi, w, z) \left| \phi' \left(\frac{z}{t} \right) \right| w^{2\lambda} dw z^{2\lambda} dz \\ &\lesssim t^{-2\lambda-3} y^{-2\lambda} |y - \tilde{y}| \int_0^\infty \left| \phi' \left(\frac{z}{t} \right) \right| z^{2\lambda} dz \\ &\lesssim t^{-2} y^{-2\lambda} |y - \tilde{y}| \int_0^\infty |\phi'(z)| z^{2\lambda} dz \\ &\lesssim t^{-2} y^{-2\lambda} |y - \tilde{y}| \\ &\lesssim \frac{1}{m_\lambda(I(y, t)) + m_\lambda(I(y, |x - y|))} \frac{|y - y'|}{t}. \end{aligned}$$

Case (iii) $8t < y$ and $|x - y| \geq y/2$. In this case, by $x < y + t$, we have $x \leq y/2$. Moreover,

$$\xi > \tilde{y} = y - (y - \tilde{y}) > y - \frac{t + y - x}{2} = \frac{y + x - t}{2} > x,$$

which implies that for any $w \in (0, x)$ and $\theta \in (0, \pi)$,

$$\sqrt{w^2 + \xi^2 - 2w\xi \cos \theta} > \xi - w > \xi - x > \frac{y - x - t}{2} \geq \frac{y}{8} \geq t.$$

This together with $\text{supp}(\phi) \subset (0, 1)$ shows that $\text{I} = 0$.

Combining the cases above we conclude that (4.8) holds, which implies (K_{ii}) .

Finally, we show that (K_{iii}) holds. Indeed, by (4.5) together with (4.4) and $\phi \in C_c^\infty(\mathbb{R}_+)$, we conclude that

$$\int_0^\infty \psi(t, x, y) dm_\lambda(y)$$

$$\begin{aligned}
&= -t^{-1}x^{-2\lambda} \int_0^\infty \int_0^\infty \int_0^x D(w, y, z) \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] w^{2\lambda} dw dz dm_\lambda(y) \\
&= -t^{-1}x^{-2\lambda} \int_0^\infty \int_0^\infty \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] dz w^{2\lambda} dw \\
&= -\frac{x}{(2\lambda+1)t} \int_0^\infty \partial_z \left[\left(\frac{z}{t} \right)^{2\lambda+1} \phi \left(\frac{z}{t} \right) \right] dz = 0.
\end{aligned}$$

This shows (K_{iii}), and hence finishes the proof of Lemma 4.1. \square

Lemma 4.2. *Let $p \in ((2\lambda+1)/(2\lambda+2), 1]$ and ϕ be as in Lemma 4.1. Then there exists a positive constant C such that for any $f, g \in L^2(\mathbb{R}_+, dm_\lambda)$ with $u(t, x) := P_t^{[\lambda]} f(x)$ satisfying $\sup_{|y-x|<t} |u(t, y)| \in L^p(\mathbb{R}_+, dm_\lambda(x))$,*

$$\begin{aligned}
\text{I} &:= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t,x} u(t, x)|^2 |\phi_t \#_\lambda g(x)|^2 t dm_\lambda(x) dt \\
&\leq C \left[\int_{\mathbb{R}_+} [f(x)]^2 [g(x)]^2 dm_\lambda(x) + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |Q_t(g)(x)|^2 \frac{dm_\lambda(x) dt}{t} \right],
\end{aligned}$$

where $Q_t(g)(x) := (t\partial_t(\phi_t \#_\lambda g)(x), t\partial_x(\phi_t \#_\lambda g)(x), \psi(g)(t, x))$ is a vector-valued function, with $\psi(g)(t, x)$ as obtained in Lemma 4.1.

Proof. First, we claim that $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$. Indeed, observe that for any $x, y, t \in \mathbb{R}_+$ with $|y-x|<t$,

$$|u(t, x)| = \left| P_t^{[\lambda]} f(x) \right| \leq \sup_{|y-x|<t} |u(t, y)|.$$

Since $\sup_{|x-y|<t} |u(t, y)| \in L^p(\mathbb{R}_+, dm_\lambda)$, we have that

$$\begin{aligned}
\left| P_t^{[\lambda]} f(x) \right|^p &\leq \frac{1}{m_\lambda(I(x, t))} \int_{I(x, t)} \left| \sup_{|x-y|<t} |u(t, y)| \right|^p dm_\lambda(y) \\
&\leq \frac{1}{m_\lambda(I(x, t))} \left\| \sup_{|x-y|<t} |u(t, y)| \right\|_{L^p(\mathbb{R}_+, dm_\lambda)}^p.
\end{aligned}$$

This means that $u(t, x) \rightarrow 0$ as $t \rightarrow \infty$ and the claim follows.

We now claim that

$$(4.9) \quad 2|\nabla_{t,x} u(t, x)|^2 = \triangle_{t,x}(u^2(t, x)).$$

In fact, recall that u satisfies the equation (1.1). We then see that

$$\triangle_{t,x}(u^2(t, x)) = \partial_t^2 u^2 + \partial_x^2 u^2 + \frac{2\lambda}{x} \partial_x u^2 = 2[(\partial_t u)^2 + (\partial_x u)^2] = 2|\nabla_{t,x} u(t, x)|^2.$$

This implies claim (4.9).

From the claim (4.9) and integration by parts, we deduce that

$$\begin{aligned}
2\text{I} &= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \triangle_{t,x}(u^2(t, x)) |\phi_t \#_\lambda g(x)|^2 t dm_\lambda(x) dt \\
&= - \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \mathcal{D}^* \mathcal{D}(u^2(t, x)) (\phi_t \#_\lambda g(x))^2 t dm_\lambda(x) dt \\
&\quad + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \partial_t^2(u^2(t, x)) [t(\phi_t \#_\lambda g(x))^2] dm_\lambda(x) dt \\
&= - \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \partial_x(u^2(t, x)) \partial_x(\phi_t \#_\lambda g(x))^2 t dm_\lambda(x) dt
\end{aligned}$$

$$\begin{aligned}
& - \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \partial_t(u^2(t, x)) \partial_t [t(\phi_t \#_\lambda g(x))^2] dm_\lambda(x) dt \\
& = -4 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(t, x) \partial_x u(t, x) \phi_t \#_\lambda g(x) \partial_x(\phi_t \#_\lambda g(x)) t dm_\lambda(x) dt \\
& \quad - 2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(t, x) \partial_t(u(t, x)) (\phi_t \#_\lambda g(x))^2 dm_\lambda(x) dt \\
& \quad - 4 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(t, x) \partial_t(u(t, x)) t(\phi_t \#_\lambda g(x)) \partial_t(\phi_t \#_\lambda g(x)) dm_\lambda(x) dt \\
& = -4 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(t, x) \nabla_{t, x} u(t, x) \cdot t(\phi_t \#_\lambda g(x)) \nabla_{t, x}(\phi_t \#_\lambda g(x)) dm_\lambda(x) dt \\
& \quad - \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \partial_t(u^2(t, x)) (\phi_t \#_\lambda g(x))^2 dm_\lambda(x) dt \\
& =: A + B.
\end{aligned}$$

For the term A, using Hölder's inequality and Cauchy's inequality, we obtain that

$$\begin{aligned}
(4.10) \quad |A| & \leq 4 \left[\iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) |\nabla_{t, x}(\phi_t \#_\lambda g(x))|^2 t dm_\lambda(x) dt \right]^{\frac{1}{2}} \\
& \quad \times \left[\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t, x} u(t, x)|^2 |\phi_t \#_\lambda g(x)|^2 t dm_\lambda(x) dt \right]^{\frac{1}{2}} \\
& \leq C \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) |\nabla_{t, x}(\phi_t \#_\lambda g(x))|^2 t dm_\lambda(x) dt \\
& \quad + \frac{1}{8} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t, x} u(t, x)|^2 |\phi_t \#_\lambda g(x)|^2 t dm_\lambda(x) dt \\
& = C \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) |t \nabla_{t, x}(\phi_t \#_\lambda g(x))|^2 \frac{dm_\lambda(x) dt}{t} + \frac{1}{8}.
\end{aligned}$$

For term B, from integration by parts, we have

$$\begin{aligned}
B & = - \int_{\mathbb{R}_+} u^2(t, x) (\phi_t \#_\lambda g(x))^2 \Big|_{t=0}^{t=\infty} dm_\lambda(x) + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \partial_t(\phi_t \#_\lambda g(x))^2 dm_\lambda(x) dt \\
& =: B_1 + B_2.
\end{aligned}$$

It is easy to see that

$$(4.11) \quad B_1 \sim \int_{\mathbb{R}_+} f(x)^2 g(x)^2 dm_\lambda(x).$$

For B_2 , we get that

$$B_2 = 2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \phi_t \#_\lambda g(x) \partial_t(\phi_t \#_\lambda g(x)) dm_\lambda(x) dt.$$

Then, using Lemma 4.1, for the function ϕ , there exists a function $\psi(t, x, y)$ such that ψ satisfies the equation (4.1) and ψ satisfies all properties listed in (ii)–(v) in Lemma 4.1. Hence, we get that

$$\begin{aligned}
B_2 & = 2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \phi_t \#_\lambda g(x) \mathcal{D}^*(\psi(g)(t, x)) dm_\lambda(x) dt \\
& = 2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \phi_t \#_\lambda g(x) \partial_x(\psi(g))(t, x) x^{2\lambda} dx dt
\end{aligned}$$

$$\begin{aligned}
& + 2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \phi_t \sharp_\lambda g(x) \frac{2\lambda}{x} \psi(g)(t, x) dm_\lambda(x) dt \\
& =: 2\mathcal{B}_{21} + 2\mathcal{B}_{22}.
\end{aligned}$$

For \mathcal{B}_{21} , integration by parts, gives

$$\begin{aligned}
\mathcal{B}_{21} &= - \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \partial_x \left(u^2(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \right) \psi(g)(t, x) dx dt \\
&+ \int_{\mathbb{R}_+} u^2(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \psi(g)(t, x) \Big|_{x=0}^{x=\infty} dt \\
&= -2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(t, x) \partial_x u(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \psi(g)(t, x) dx dt \\
&- \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \partial_x (\phi_t \sharp_\lambda g(x)) x^{2\lambda} \psi(g)(t, x) dx dt \\
&- \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \phi_t \sharp_\lambda g(x) 2\lambda x^{2\lambda-1} \psi(g)(t, x) dx dt \\
&+ \int_{\mathbb{R}_+} u^2(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \psi(g)(t, x) \Big|_{x=0}^{x=\infty} dt,
\end{aligned}$$

where the third term on the right-hand side equals $-\mathcal{B}_{22}$. Hence,

$$\begin{aligned}
\mathcal{B}_2 &= -2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u(t, x) \partial_x u(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \psi(g)(t, x) dx dt \\
&- \iint_{\mathbb{R}_+ \times \mathbb{R}_+} u^2(t, x) \partial_x (\phi_t \sharp_\lambda g(x)) x^{2\lambda} \psi(g)(t, x) dx dt \\
&+ \int_{\mathbb{R}_+} u^2(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \psi(g)(t, x) \Big|_{x=0}^{x=\infty} dt, \\
&=: \mathcal{B}_{21} + \mathcal{B}_{22} + \mathcal{B}_{23}.
\end{aligned}$$

For the term \mathcal{B}_{23} , we first note that

$$\begin{aligned}
\mathcal{B}_{23} &= - \int_{\mathbb{R}_+} \lim_{x \rightarrow 0} \left(u^2(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \psi_t \sharp_\lambda g(x) \right) dt \\
&+ \int_{\mathbb{R}_+} \lim_{x \rightarrow \infty} \left(u^2(t, x) \phi_t \sharp_\lambda g(x) x^{2\lambda} \psi_t \sharp_\lambda g(x) \right) dt \\
&=: \mathcal{B}_{231} + \mathcal{B}_{232}.
\end{aligned}$$

We claim that

$$(4.12) \quad \mathcal{B}_{231} = \mathcal{B}_{232} = 0.$$

We first consider \mathcal{B}_{231} . Letting $x \rightarrow 0+$ and applying Hölder's inequality, we see that

$$\begin{aligned}
\lim_{x \rightarrow 0} |u(t, x)| &\leq \lim_{x \rightarrow 0} \left[\int_0^\infty P_t^{[\lambda]}(x, y)^2 y^{2\lambda} dy \right]^{\frac{1}{2}} \|f\|_{L^2(\mathbb{R}_+, dm_\lambda)} \\
&= \|f\|_{L^2(\mathbb{R}_+, dm_\lambda)} \frac{2\lambda t}{\pi} \left\{ \int_0^\infty \left[\int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{(y^2 + t^2)^{\lambda+1}} d\theta \right]^2 y^{2\lambda} dy \right\}^{\frac{1}{2}} \\
&\lesssim t \|f\|_{L^2(\mathbb{R}_+, dm_\lambda)} \left[\int_0^\infty \frac{y^{2\lambda}}{(y^2 + t^2)^{2\lambda+2}} dy \right]^{\frac{1}{2}} \\
&\lesssim t^{-\lambda-\frac{1}{2}} \|f\|_{L^2(\mathbb{R}_+, dm_\lambda)},
\end{aligned}$$

and

$$\begin{aligned}
& \lim_{x \rightarrow 0} |\phi_t \#_\lambda g(x)| \\
& \lesssim \lim_{x \rightarrow 0} \left| \int_0^\infty \int_0^\pi \left[t^{-2\lambda-1} \phi \left(\frac{\sqrt{x^2 + y^2 - 2xy \cos \theta}}{t} \right) (\sin \theta)^{2\lambda-1} d\theta \right]^2 y^{2\lambda} dy \right|^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}_+, dm_\lambda)} \\
& \sim \left| \int_0^\infty \left[t^{-2\lambda-1} \phi \left(\frac{y}{t} \right) \right]^2 y^{2\lambda} dy \right|^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}_+, dm_\lambda)} \\
& \sim t^{-\lambda-\frac{1}{2}} \|\phi\|_{L^2(\mathbb{R}_+, dm_\lambda)} \|g\|_{L^2(\mathbb{R}_+, dm_\lambda)}.
\end{aligned}$$

Moreover, by (4.2), (K_i) and (4.3), we see that

$$\|\psi\|_{L^1(\mathbb{R}_+, dm_\lambda)} \lesssim 1 \text{ and } |\psi(t, x, y)| \lesssim \frac{x}{t^{2\lambda+2}}.$$

Thus, we have that

$$\begin{aligned}
\lim_{x \rightarrow 0} |\psi(g)(t, x)| & \leq \lim_{x \rightarrow 0} \|\psi\|_{L^1(\mathbb{R}_+, dm_\lambda)}^{\frac{1}{2}} \left[\int_0^\infty |\psi(t, x, y)| |g(y)|^2 dm_\lambda(y) \right]^{\frac{1}{2}} \\
& \lesssim \lim_{x \rightarrow 0} x^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}_+, dm_\lambda)} = 0.
\end{aligned}$$

Therefore, we obtain that

$$\lim_{x \rightarrow 0} \left| u^2(t, x) \phi_t \#_\lambda g(x) x^{2\lambda} \psi(g)(t, x) \right| = 0,$$

which gives that $B_{231} = 0$. Next we verify the term B_{232} . Note that

$$\left| P_t^{[\lambda]}(x, y) \right| \lesssim \frac{1}{m_\lambda(I(x, |x-y|+t))} \frac{t}{|x-y|+t}.$$

Then by Hölder's inequality, we have that

$$\begin{aligned}
|u(t, x)|^2 & \lesssim \|f\|_{L^2(\mathbb{R}_+, dm_\lambda)}^2 \sum_{k=0}^\infty 2^{-2k} \int_{|x-y| < 2^k t} \frac{1}{[m_\lambda(I(x, 2^{k-1}t))]^2} dm_\lambda(y) \\
& \lesssim \|f\|_{L^2(\mathbb{R}_+, dm_\lambda)}^2 \frac{1}{x^{2\lambda} t}.
\end{aligned}$$

Moreover, from (4.6) and Hölder inequality, we deduce that

$$|\psi(g)(t, x)| \leq \|\psi\|_{L^1(\mathbb{R}_+, dm_\lambda)}^{\frac{1}{2}} \left[\int_0^\infty |\psi(t, x, y)| |g(y)|^2 dm_\lambda(y) \right]^{\frac{1}{2}} \lesssim x^{-\lambda} t^{-1/2} \|g\|_{L^2(\mathbb{R}_+, dm_\lambda)}.$$

By these and the fact that

$$|\phi_t \#_\lambda g(x)| \leq \|\phi\|_{L^2(\mathbb{R}_+, dm_\lambda)} \|g\|_{L^2(\mathbb{R}_+, dm_\lambda)},$$

we obtain that

$$\lim_{x \rightarrow \infty} \left| u^2(t, x) \phi_t \#_\lambda g(x) x^{2\lambda} \psi(g)(t, x) \right| = 0,$$

which implies that

$$B_{232} = 0.$$

Hence, the claim (4.12) holds.

Similar to the estimate for the term A, as for the term B_{21} , using Hölder's inequality and Cauchy's inequality, we obtain that

$$(4.13) \quad B_{21} \leq \frac{1}{8} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t,x} u(t, x)|^2 |\phi_t \#_\lambda g(x)|^2 t dm_\lambda(x) dt$$

$$\begin{aligned}
& + C \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |\psi(g)(t, x)|^2 \frac{dm_\lambda(x) dt}{t} \\
& = \frac{1}{8} + C \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |\psi(g)(t, x)|^2 \frac{dm_\lambda(x) dt}{t}.
\end{aligned}$$

Again, for the term B_{22} we have

$$B_{22} \lesssim \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |t \partial_x (\phi_t \#_\lambda g(x))|^2 \frac{dm_\lambda(x) dt}{t} + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |\psi(g)(t, x)|^2 \frac{dm_\lambda(x) dt}{t}.$$

Combining the estimates of A and B, (4.10), (4.11), (4.13), (4.14) and (4.12), and by moving the two terms $\frac{1}{8}$ to the left-hand side, we obtain that

$$\begin{aligned}
I & \lesssim \int_{\mathbb{R}_+} f(x)^2 g(x)^2 dm_\lambda(x) + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |t \partial_x (\phi_t \#_\lambda g(x))|^2 \frac{dm_\lambda(x) dt}{t} \\
& \quad + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |t \partial_t (\phi_t \#_\lambda g(x))|^2 \frac{dm_\lambda(x) dt}{t} + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |\psi(g)(t, x)|^2 \frac{dm_\lambda(x) dt}{t}.
\end{aligned}$$

We now define

$$Q_t(g)(x) := (t \partial_t (\phi_t \#_\lambda g(x)), t \partial_x (\phi_t \#_\lambda g(x)), \psi(g)(t, x)).$$

Then we have

$$I \lesssim \int_{\mathbb{R}_+} f(x)^2 g(x)^2 dm_\lambda(x) + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t, x)|^2 |Q_t(g)(x)|^2 \frac{dm_\lambda(x) dt}{t}.$$

This finishes the proof of Lemma 4.2. \square

Next we have the following result for the product case, which follows from the iteration of Lemma 4.2. Before stating our next Lemma, we introduce the notation $\phi_{t_1} \#_{\lambda, 1} g(x_1, x_2)$, $\phi_{t_2} \#_{\lambda, 2} g(x_1, x_2)$ and $\phi_{t_1} \phi_{t_2} \#_{\lambda, 1, 2} g(x_1, x_2)$ to denote the convolution with respect to the first, second and both variables, respectively.

Lemma 4.3. *Let $u(t_1, t_2, x_1, x_2) := P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f(x_1, x_2)$ and ϕ be a smooth function as in Lemma 4.1. Then for $f, g \in L^2(\mathbb{R}_\lambda)$, there exists a positive constant C such that*

$$\begin{aligned}
\tilde{I} & := \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\nabla_{t_1, x_1} \nabla_{t_2, x_2} u(t_1, t_2, x_1, x_2)|^2 \\
& \quad \times |\phi_{t_1} \phi_{t_2} \#_{\lambda, 1, 2} g(x_1, x_2)|^2 t_1 t_2 d\mu_\lambda(x_1, x_2) dt_1 dt_2 \\
& \leq C \left\{ \iint_{\mathbb{R}_+ \times \mathbb{R}_+} [f(x_1, x_2)]^2 [g(x_1, x_2)]^2 d\mu_\lambda(x_1, x_2) \right. \\
& \quad + \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| P_{t_2}^{[\lambda]} f(x_1, x_2) \right|^2 \left[Q_{t_2}^{(2)}(g)(x_1, x_2) \right]^2 \frac{dm_\lambda(x_2) dt_2}{t_2} dm_\lambda(x_1) \\
& \quad + \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| P_{t_1}^{[\lambda]} f(x_1, x_2) \right|^2 \left[Q_{t_1}^{(1)}(g)(x_1, x_2) \right]^2 \frac{dm_\lambda(x_1) dt_1}{t_1} dm_\lambda(x_2) \\
& \quad \left. + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t_1, t_2, x_1, x_2)|^2 \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(x_1, x_2) \right|^2 \frac{d\mu_\lambda(x_1, x_2) dt_1 dt_2}{t_1 t_2} \right\}.
\end{aligned}$$

Here the operator $Q_{t_1}^{(1)}$ is defined as

$$Q_{t_1}^{(1)}(g)(x_1, x_2) := (t_1 \partial_{t_1} (\phi_{t_1} \#_{\lambda, 1} g)(x_1, x_2), t_1 \partial_{x_1} (\phi_{t_1} \#_{\lambda, 1} g)(x_1, x_2), \psi(g(\cdot, x_2))(t_1, x_1)),$$

where

$$\psi(g(\cdot, x_2))(t_1, x_1) = \int_0^\infty \psi(t_1, x_1, y_1) g(y_1, x_2) dm_\lambda(y_1)$$

is obtained from Lemma 4.1. The definition of $Q_{t_2}^{(2)}(g)(x_1, x_2)$ is similar.

Proof. By Lemma 4.2 for t_1 and x_1 and the conservation property of Poisson semigroup, we have that

$$\begin{aligned}
\tilde{I} &= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \nabla_{t_1, x_1} P_{t_1}^{[\lambda]} \left(\nabla_{t_2, x_2} P_{t_2}^{[\lambda]} f(\cdot, t_2, \cdot, x_2) \right) (t_1, x_1) \right|^2 \\
&\quad \times \left| \phi_{t_1} \sharp_{\lambda, 1} \left(\phi_{t_2} \sharp_{\lambda, 2} g(\cdot, x_2) \right) (x_1) \right|^2 t_1 t_2 d\mu_\lambda(x_1, x_2) dt_1 dt_2 \\
&\lesssim \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \int_{\mathbb{R}_+} \left| \nabla_{t_2, x_2} \left(P_{t_2}^{[\lambda]} f \right) (x_1, x_2) \right|^2 |(\phi_{t_2} \sharp_{\lambda, 2} g)(x_1, x_2)|^2 dm_\lambda(x_1) t_2 dm_\lambda(x_2) dt_2 \\
&\quad + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \nabla_{t_2, x_2} \left(P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \right) (x_1, x_2) \right|^2 \\
&\quad \times \left[Q_{t_1}^{(1)}(\phi_{t_2} \sharp_{\lambda, 2} g(\cdot, x_2))(x_1) \right]^2 \frac{dm_\lambda(x_1) dt_1}{t_1} t_2 dm_\lambda(x_2) dt_2 \\
&= \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \nabla_{t_2, x_2} \left(P_{t_2}^{[\lambda]} f(x_1, \cdot) \right) (x_2) \right|^2 |(\phi_{t_2} \sharp_{\lambda, 2} g(x_1, \cdot))(x_2)|^2 t_2 dm_\lambda(x_2) dt_2 dm_\lambda(x_1) \\
&\quad + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| \nabla_{t_2, x_2} P_{t_2}^{[\lambda]} \left(P_{t_1}^{[\lambda]} f(x_1, \cdot) \right) (x_1, x_2) \right|^2 \\
&\quad \times \left[\phi_{t_2} \sharp_{\lambda, 2} (Q_{t_1}^{(1)}(g(\cdot, \cdot))(x_1))(x_2) \right]^2 \frac{dm_\lambda(x_1) dt_1}{t_1} t_2 dm_\lambda(x_2) dt_2 \\
&=: \tilde{I}_1 + \tilde{I}_2,
\end{aligned}$$

where in the last but second equality, we use the fact that the order of $\sharp_{\lambda, 2}$ and $Q_{t_1}^{(1)}$ can be changed since they are acting on different variables and $Q_{t_1}^{(1)}$ is a linear integral operator.

Now we apply Lemma 4.2 to \tilde{I}_1 and see that

$$\begin{aligned}
\tilde{I}_1 &\lesssim \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |f(x_1, x_2)|^2 |g(x_1, x_2)|^2 dm_\lambda(x_1) dm_\lambda(x_2) \\
&\quad + \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| P_{t_2}^{[\lambda]} f(x_1, x_2) \right|^2 \left[Q_{t_2}^{(2)}(g)(x_1, x_2) \right]^2 \frac{dm_\lambda(x_2) dt_2}{t_2} dm_\lambda(x_1).
\end{aligned}$$

Similarly, another application of Lemma 4.2 yields that

$$\begin{aligned}
\tilde{I}_2 &\lesssim \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| P_{t_1}^{[\lambda]} f(x_1, x_2) \right|^2 \left[Q_{t_1}^{(1)}(g)(x_1, x_2) \right]^2 dm_\lambda(x_2) \frac{dm_\lambda(x_1) dt_1}{t_1} \\
&\quad + \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t_1, t_2, x_1, x_2)|^2 \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(x_1, x_2) \right|^2 \frac{d\mu_\lambda(x_1, x_2) dt_1 dt_2}{t_1 t_2}.
\end{aligned}$$

This finishes the proof of Lemma 4.3. \square

Proof of $\|f\|_{H_{S_u}^p(\mathbb{R}_\lambda)} \leq C \|f\|_{H_{\mathcal{N}_P}^p(\mathbb{R}_\lambda)}$.

For any $\alpha > 0$ and $f \in L^2(\mathbb{R}_\lambda)$ satisfying $\mathcal{N}_P(f) \in L^p(\mathbb{R}_\lambda)$, we define

$$\mathcal{A}(\alpha) := \left\{ (x, y) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{M}_S(\chi_{\mathcal{N}_P(f) > \alpha})(x, y) < \frac{1}{200} \frac{1}{2^{4\lambda+2}} \right\},$$

where \mathcal{M}_S is the strong maximal function defined in (3.4). We first claim that

$$\begin{aligned}
(4.14) \quad &\iint_{\mathcal{A}(\alpha)} S_u^2(f)(x_1, x_2) d\mu_\lambda(x_1, x_2) \\
&\leq \iiint_{R^*} |t_1 t_2 \nabla_{t_1, y_1} \nabla_{t_2, y_2} u(t_1, t_2, y_1, y_2)|^2 \frac{d\mu_\lambda(y_1, y_2) dt_1 dt_2}{t_1 t_2},
\end{aligned}$$

where for $t_1, t_2, y_1, y_2 \in \mathbb{R}_+$, $R(y_1, y_2, t_1, t_2) := I(y_1, t_1) \times I(y_2, t_2)$ and

$$R^* := \left\{ (y_1, y_2, t_1, t_2) : \frac{\mu_\lambda(\{\mathcal{N}_P(f) > \alpha\} \cap R(y_1, y_2, t_1, t_2))}{\mu_\lambda(R(y_1, y_2, t_1, t_2))} < \frac{1}{200} \frac{1}{2^{4\lambda+2}} \right\}.$$

Indeed, observe that

$$\begin{aligned} & \iint_{\mathcal{A}(\alpha)} S_u^2(f)(x_1, x_2) d\mu_\lambda(x_1, x_2) \\ & \leq \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathcal{A}(\alpha) \cap R(y_1, y_2, t_1, t_2)} \\ & \quad \left| \nabla_{t_1, y_1} \nabla_{t_2, y_2} u(t_1, y_1, t_2, y_2) \right|^2 t_1 t_2 \frac{d\mu_\lambda(x_1, x_2)}{m_\lambda(I(x_1, t_1)) m_\lambda(I(x_2, t_2))} d\mu_\lambda(y_1, y_2) dt_1 dt_2. \end{aligned}$$

For any $(y_1, y_2, t_1, t_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ with $R(y_1, y_2, t_1, t_2) \cap \mathcal{A}(\alpha) \neq \emptyset$, there exists some $(x_1, x_2) \in R(y_1, y_2, t_1, t_2) \cap \mathcal{A}(\alpha)$ such that

$$\mathcal{M}_S(\chi_{\mathcal{N}_P(f) > \alpha})(x_1, x_2) < \frac{1}{200} \frac{1}{2^{4\lambda+2}}.$$

Hence we have

$$\frac{\mu_\lambda(\{\mathcal{N}_P(f) > \alpha\} \cap R(y_1, y_2, t_1, t_2))}{\mu_\lambda(R(y_1, y_2, t_1, t_2))} < \frac{1}{200} \frac{1}{2^{4\lambda+2}}.$$

Then by the fact that for any $y_1 \in I(x_1, t_1)$ and $y_2 \in I(x_2, t_2)$,

$$m_\lambda(I(x_1, t_1)) \sim m_\lambda(I(y_1, t_1)) \quad \text{and} \quad m_\lambda(I(x_2, t_2)) \sim m_\lambda(I(y_2, t_2)),$$

we have (4.14) and the claim holds.

Let $g(x, y) := \chi_{\{\mathcal{N}_P(f) \leq \alpha\}}(x, y)$ and $\phi \in C_c^\infty(\mathbb{R}_+)$ such that $\text{supp}(\phi) \subset (0, 1)$, $\phi \equiv 1$ on $(0, 1/2]$ and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}_+$. Then for $(x_1, x_2, t_1, t_2) \in R^*$, we have

$$\begin{aligned} (4.15) \quad & \phi_{t_1} \phi_{t_2} \sharp_{\lambda, 1, 2} g(x_1, x_2) \\ & = \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \tau_{x_1}^{[\lambda]} \tau_{x_2}^{[\lambda]}(\phi_{t_1} \phi_{t_2})(y_1, y_2) \chi_{\{\mathcal{N}_P(f) \leq \alpha\}}(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2) \\ & = \iint_{\{\mathcal{N}_P(f) \leq \alpha\} \cap R(x_1, x_2, t_1, t_2)} \tau_{x_1}^{[\lambda]} \tau_{x_2}^{[\lambda]}(\phi_{t_1} \phi_{t_2})(y_1, y_2) dm_\lambda(y_1) dm_\lambda(y_2) \\ & \geq \iint_{\{\mathcal{N}_P(f) \leq \alpha\} \cap R(x_1, x_2, t_1/2, t_2/2)} dm_\lambda(y_1) dm_\lambda(y_2) \gtrsim 1, \end{aligned}$$

where the last inequality follows from the fact that

$$\begin{aligned} & \mu_\lambda(\{(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{N}_P(f)(y_1, y_2) \leq \alpha\} \cap R(x_1, x_2, t_1/2, t_2/2)) \\ & \geq \mu_\lambda(\{(y_1, y_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{N}_P(f)(y_1, y_2) \leq \alpha\} \cap R(x_1, x_2, t_1, t_2)) \\ & \quad - \mu_\lambda(R(x_1, x_2, t_1, t_2) \setminus R(x_1, x_2, t_1/2, t_2/2)) \\ & \geq \mu_\lambda(R(x_1, x_2, t_1/2, t_2/2)) - \frac{1}{200} \frac{1}{2^{4\lambda+2}} \mu_\lambda(R(x_1, x_2, t_1, t_2)) \\ & \gtrsim \mu_\lambda(R(x_1, x_2, t_1, t_2)). \end{aligned}$$

Combining (4.14) and (4.15), and then using Lemma 4.3, we have

$$\begin{aligned} & \iint_{\mathcal{A}(\alpha)} S_u^2(f)(x_1, x_2) d\mu_\lambda(x_1, x_2) \\ & \lesssim \iiint_{R^*} \left| t_1 t_2 \nabla_{t_1, y_1} \nabla_{t_2, y_2} u(t_1, t_2, y_1, y_2) \right|^2 |\phi_{t_1} \phi_{t_2} \sharp_{\lambda, 1, 2} g(y_1, y_2)|^2 \frac{d\mu_\lambda(y_1, y_2) dt_1 dt_2}{t_1 t_2} \\ & \lesssim \left\{ \iint_{\mathbb{R}_+ \times \mathbb{R}_+} [f(x_1, x_2)]^2 [g(x_1, x_2)]^2 d\mu_\lambda(x_1, x_2) \right\} \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| P_{t_2}^{[\lambda]} f(x_1, x_2) \right|^2 \left| Q_{t_2}^{(2)}(g)(x_1, x_2) \right|^2 \frac{dm_\lambda(x_2) dt_2}{t_2} dm_\lambda(x_1) \\
& + \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| P_{t_1}^{[\lambda]} f(x_1, x_2) \right|^2 \left| Q_{t_1}^{(1)}(g)(x_1, x_2) \right|^2 \frac{dm_\lambda(x_1) dt_1}{t_1} dm_\lambda(x_2) \\
& + \left\{ \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t_1, t_2, x_1, x_2)|^2 \left| Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(x_1, x_2) \right|^2 \frac{d\mu_\lambda(x_1, x_2) dt_1 dt_2}{t_1 t_2} \right\} \\
& =: \text{I} + \text{II} + \text{III} + \text{IV}.
\end{aligned}$$

For the term I, we have

$$\text{I} = \iint_{\{\mathcal{N}_P(f) \leq \alpha\}} f^2(y_1, y_2) d\mu_\lambda(y_1, y_2) \leq \iint_{\{\mathcal{N}_P(f) \leq \alpha\}} |\mathcal{N}_P(f)(y_1, y_2)|^2 d\mu_\lambda(y_1, y_2).$$

For the term II, we claim that: if $|Q_{t_2}^{(2)}(g)(x_1, x_2)| \neq 0$, then there exists some w_2 such that $(x_1, w_2) \in \{\mathcal{N}_P(f) \leq \alpha\}$, and satisfies $|x_2 - w_2| < t_2$. To see this, recall that

$$Q_{t_2}^{(2)}(g)(x_1, x_2) := (t_2 \partial_{t_2}(\phi_{t_2} \sharp_{\lambda, 2g}(x_1, x_2)), t_2 \partial_{x_2}(\phi_{t_2} \sharp_{\lambda, 2g}(x_1, x_2)), \psi(g(x_1, \cdot))(t_2, x_2)),$$

where

$$\psi(g(x_1, \cdot))(t_2, x_2) = \int_0^\infty \psi(t_2, x_2, y_2) g(x_1, y_2) dm_\lambda(y_2)$$

is obtained from Lemma 4.1. Hence, if $|Q_{t_2}^{(2)}(g)(x_1, x_2)| \neq 0$, then we have that one of the three terms $\partial_{t_2}(\phi_{t_2} \sharp_{\lambda, 2g})(x_1, x_2)$, $\partial_{x_2}(\phi_{t_2} \sharp_{\lambda, 2g})(x_1, x_2)$, $\psi(g(x_1, \cdot))(t_2, x_2)$ must be non-zero. Hence, there must be some w_2 such that (x_1, w_2) is in the support of the function g , and satisfies $|x_2 - w_2| < t_2$. This implies that the claim holds.

Then we get that $|P_{t_2}^{[\lambda]} f(x_1, x_2)| \leq \alpha$. As a consequence,

$$\begin{aligned}
(4.16) \quad \text{II} & \leq \alpha^2 \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| Q_{t_2}^{(2)}(g)(x_1, x_2) \right|^2 \frac{dm_\lambda(x_2) dt_2}{t_2} dm_\lambda(x_1) \\
& = \alpha^2 \int_{\mathbb{R}_+} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| Q_{t_2}^{(2)}(1 - g)(x_1, x_2) \right|^2 \frac{dm_\lambda(x_2) dt_2}{t_2} dm_\lambda(x_1) \\
& \lesssim \alpha^2 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |1 - g(x_1, x_2)|^2 d\mu_\lambda(x_1, x_2) \\
& \lesssim \alpha^2 \mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{N}_P(f)(x_1, x_2) > \alpha\}),
\end{aligned}$$

where the second inequality follows from the L^2 boundedness of the Littlewood–Paley square function estimate, i.e., (2.5) of Theorem 2.12. For the term III, symmetrically, we can obtain the same estimate as term II.

For the term IV, if $Q_{t_1}^{(1)} Q_{t_2}^{(2)}(g)(x_1, x_2) \neq 0$, then there exist some (w_1, w_2) such that $(w_1, w_2) \in \{\mathcal{N}_P(f) \leq \alpha\}$ and $|x_1 - w_1| < t_1$ and $|x_2 - w_2| < t_2$. Hence $|u(t_1, t_2, x_1, x_2)| \leq \alpha$. Following the same routine of (4.16), and using the L^2 boundedness of the product Littlewood–Paley square function estimate, i.e., (2.6) of Theorem 2.12, we have

$$\text{IV} \lesssim \alpha^2 \mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{N}_P(f)(x_1, x_2) > \alpha\}).$$

Combining the four terms above, we have

$$\begin{aligned}
(4.17) \quad & \iint_{\mathcal{A}(\alpha)} S_u^2(f)(x_1, x_2) d\mu_\lambda(x_1, x_2) \\
& \lesssim \alpha^2 \mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{N}_P(f)(x_1, x_2) > \alpha\}) \\
& \quad + \iint_{\{\mathcal{N}_P(f) \leq \alpha\}} |\mathcal{N}_P(f)(x_1, x_2)|^2 d\mu_\lambda(x_1, x_2).
\end{aligned}$$

By the $L^2(\mathbb{R}_\lambda)$ -boundedness of the strong maximal function \mathcal{M}_S , we have

$$\begin{aligned}
 (4.18) \quad \mu_\lambda(\mathbb{R}_+ \times \mathbb{R}_+ \setminus \mathcal{A}(\alpha)) &\lesssim \iint_{\mathbb{R}_+ \times \mathbb{R}_+} [\mathcal{M}_S(\chi_{\{\mathcal{N}_P(f) > \alpha\}})(x_1, x_2)]^2 d\mu_\lambda(x_1, x_2) \\
 &\lesssim \iint_{\mathbb{R}_+ \times \mathbb{R}_+} [\chi_{\{\mathcal{N}_P(f) > \alpha\}}(x_1, x_2)]^2 d\mu_\lambda(x_1, x_2) \\
 &= \mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{N}_P(f)(x_1, x_2) > \alpha\}).
 \end{aligned}$$

Combining (4.17) and (4.18), we have

$$\begin{aligned}
 &\mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : S_u(f)(x_1, x_2) > \alpha\}) \\
 &\leq \mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : S_u(f)(x_1, x_2) > \alpha\} \cap \mathcal{A}(\alpha)) \\
 &\quad + \mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : S_u(f)(x_1, x_2) > \alpha\} \setminus \mathcal{A}(\alpha)) \\
 &\lesssim \mu_\lambda(\{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{N}_P(f)(x_1, x_2) > \alpha\}) \\
 &\quad + \frac{1}{\alpha^2} \iint_{\{\mathcal{N}_P(f) \leq \alpha\}} |\mathcal{N}_P(f)(x_1, x_2)|^2 d\mu_\lambda(x_1, x_2),
 \end{aligned}$$

which via a standard argument shows that $\|S_u(f)\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|\mathcal{N}_P(f)\|_{L^p(\mathbb{R}_\lambda)}$. \square

Step 3: $\|f\|_{H_{\mathcal{N}_P}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda)}$ for $f \in H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$.

We now define the product grand maximal functions, borrowing an idea from [YZ] in the one-parameter setting (see also [GLY1, GLY2]).

Definition 4.4. Let $\beta_1, \beta_2, \gamma_1, \gamma_2 \in (0, 1]$. For any $f \in (\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$, we define the product grand maximal function as follows: For $(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$(4.19) \quad G_{\beta_1, \beta_2, \gamma_1, \gamma_2}(f)(x_1, x_2) := \sup \{ \langle f, \varphi_1 \varphi_2 \rangle : \|\varphi_i\|_{\mathcal{G}(x_i, r_i, \beta_i, \gamma_i)} \leq 1, r_i > 0, i = 1, 2 \}.$$

By the definition of $\mathcal{N}_P f$, we have

$$\mathcal{N}_P f(x_1, x_2) = \sup_{\substack{|y_1 - x_1| < t_1 \\ |y_2 - x_2| < t_2}} \left| \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} P_{t_1}^{[\lambda]}(y_1, z_1) P_{t_2}^{[\lambda]}(y_2, z_2) f(z_1, z_2) d\mu_\lambda(z_1, z_2) \right|.$$

Next, for $i = 1, 2$, for any fixed $y_i, t_i \in \mathbb{R}_+$, the Poisson kernel $P_{t_i}^{[\lambda]}(y_i, z_i)$, as a function of z_i , satisfies the conditions (K_i) and (K_{ii}) as in Section 2.3 (see [YY]), and hence, it is a test function of the type $(y_i, t_i, 1, 1)$, with the norm $\|P_{t_i}^{[\lambda]}(y_i, \cdot)\|_{\mathcal{G}(y_i, t_i, 1, 1)} =: C_\lambda$, where C_λ is a positive constant depending only on λ (see Definition 2.2 for the test function and its norm). Hence, it is a test function of the type $(y_i, t_i, \beta_i, \gamma_i)$ for every $\beta_i, \gamma_i \in (0, 1]$ with the norm C_λ . Moreover, for any x_i with $|x_i - y_i| < t_i$, we have that $\|P_{t_i}^{[\lambda]}(y_i, \cdot)\|_{\mathcal{G}(x_i, t_i, \beta_i, \gamma_i)} \lesssim C_\lambda$, where the implicit constant is independent of x_i, t_i, β_i and γ_i .

Then, there exists a positive constant \tilde{C}_λ such that

$$\sup_{|y_1 - x_1| < t_1} \|P_{t_1}^{[\lambda]}(y_1, \cdot)\|_{\mathcal{G}(x_1, t_1, \beta_1, \gamma_1)} = \sup_{|y_2 - x_2| < t_2} \|P_{t_2}^{[\lambda]}(y_2, \cdot)\|_{\mathcal{G}(x_2, t_2, \beta_2, \gamma_2)} = \tilde{C}_\lambda.$$

We then obtain that

$$\mathcal{N}_P f(x_1, x_2) \lesssim G_{\beta_1, \beta_2, \gamma_1, \gamma_2}(f)(x_1, x_2).$$

Next we claim that

$$(4.20) \quad G_{\beta_1, \beta_2, \gamma_1, \gamma_2}(f)(x_1, x_2) \lesssim \{\mathcal{M}_1 \mathcal{M}_2(|\mathcal{R}_P(f)|^r)(x_1, x_2)\}^{\frac{1}{r}}$$

for any $r \in (\frac{2\lambda+1}{2\lambda+2}, p)$ and $f \in H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$, where \mathcal{M}_1 and \mathcal{M}_2 are as in Section 2. This implies our Step 3.

To prove (4.20), we first prove the following inequality:

$$(4.21) \quad |\langle f, \psi_1 \psi_2 \rangle| \lesssim \left[\mathcal{M}_1 \left(\mathcal{M}_2 \left(|\mathcal{R}_P(f)|^r \right) \right) (x_1, x_2) \right]^{\frac{1}{r}}$$

for any $r \in (\frac{2\lambda+1}{2\lambda+2}, p)$, $f \in H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$, $\psi_1 \in \mathring{\mathcal{G}}_1(\beta_1, \gamma_1)$ with $\|\psi_1\|_{\mathcal{G}(x_1, 2^{-\tilde{k}_1}, \beta_1, \gamma_1)} \leq 1$ and $\psi_2 \in \mathring{\mathcal{G}}_1(\beta_2, \gamma_2)$ with $\|\psi_2\|_{\mathcal{G}(x_2, 2^{-\tilde{k}_2}, \beta_2, \gamma_2)} \leq 1$.

To see this, consider the following approximations to the identity: For each $k \in \mathbb{Z}$, define the operator

$$(4.22) \quad P_k := P_{2^{-k}}^{[\lambda]}$$

with the kernel $P_k(x, y) := P_{2^{-k}}^{[\lambda]}(x, y)$. Then, it is easy to see that

$$\lim_{k \rightarrow \infty} P_k = \lim_{k \rightarrow \infty} P_{2^{-k}}^{[\lambda]} = Id \quad \text{and} \quad \lim_{k \rightarrow -\infty} P_k = \lim_{k \rightarrow -\infty} P_{2^{-k}}^{[\lambda]} = 0$$

in the sense of $L^2(\mathbb{R}_+, dm_\lambda)$. Moreover, based on size and smoothness conditions of the Poisson kernel $P_t^{[\lambda]}(x, y)$, it is direct that $P_k(x, y)$ satisfies the size and smoothness conditions as in (A_i), (A_{ii}) and (A_{iii}) in Definition 2.1 for x, y with a certain positive constant \overline{C}_λ .

Also, from (S_{iii}) in Lemma 2.9, we have that for any $k \in \mathbb{Z}$ and $x \in \mathbb{R}_+$,

$$\int_{\mathbb{R}_+} P_k(x, y) dm_\lambda(y) = \int_{\mathbb{R}_+} P_k(x, y) dm_\lambda(x) = 1.$$

Hence, $\{P_k\}_{k \in \mathbb{Z}}$ is an approximation to the identity as in Definition 2.1. Then we set $Q_k := P_k - P_{k-1}$ as the difference operator, and it is obvious that the kernel $Q_k(x, y)$ of Q_k satisfies the same size and smoothness conditions as $P_k(x, y)$ does, and

$$\int_{\mathbb{R}_+} Q_k(x, y) dm_\lambda(y) = \int_{\mathbb{R}_+} Q_k(x, y) dm_\lambda(x) = 0.$$

Now to classify the action on different variables, for $i = 1, 2$, we let $\{P_{k_i}^{(i)}\}_{k_i \in \mathbb{Z}}$ be the approximation to the identity on the i th variable as defined above, and similarly let $Q_{k_i}^{(i)}$ be the corresponding difference operator.

Then, following Theorem 2.9 in [HLLu], we now have the following Calderón's reproducing formula:

$$(4.23) \quad f(x_1, x_2) = \sum_{k_1} \sum_{I_1 \in \mathcal{X}^{k_1+N_1}} \sum_{k_2} \sum_{I_2 \in \mathcal{X}^{k_2+N_2}} m_\lambda(I_1) m_\lambda(I_2) \\ \times \tilde{Q}_{k_1}^{(1)}(x_1, x_{I_1}) \tilde{Q}_{k_2}^{(2)}(x_2, x_{I_2}) Q_{k_1}^{(1)} Q_{k_2}^{(2)}(f)(x_{I_1}, x_{I_2}),$$

where the series converges in the sense of $(\mathring{\mathcal{G}}_{1,1}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$, and for $i := 1, 2$, $\tilde{Q}_{k_i}^{(i)}$ satisfies the same size, smoothness and cancellation conditions as $Q_{k_i}^{(i)}$ does, \mathcal{X}^k is as in Section 2, and x_{I_1}, x_{I_2} are arbitrary points in the dyadic intervals I_1 and I_2 , respectively.

We now prove (4.21). To begin with, for any $f \in H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$ and $\psi_1 \in \mathring{\mathcal{G}}_1(\beta_1, \gamma_1)$ and with $\|\psi_1\|_{\mathcal{G}(x_1, 2^{-\tilde{k}_1}, \beta_1, \gamma_1)} \leq 1$ and $\psi_2 \in \mathring{\mathcal{G}}_1(\beta_2, \gamma_2)$ with $\|\psi_2\|_{\mathcal{G}(x_2, 2^{-\tilde{k}_2}, \beta_2, \gamma_2)} \leq 1$, from (4.23) we obtain that

$$\langle f, \psi_1 \psi_2 \rangle = \sum_{k_1} \sum_{I_1 \in \mathcal{X}^{k_1+N_1}} \sum_{k_2} \sum_{I_2 \in \mathcal{X}^{k_2+N_2}} m_\lambda(I_1) m_\lambda(I_2)$$

$$\times \tilde{Q}_{k_1}^{(1)}(\psi_1)(x_{I_1})\tilde{Q}_{k_2}^{(2)}(\psi_2)(x_{I_2})Q_{k_1}^{(1)}Q_{k_2}^{(2)}(f)(x_{I_1}, x_{I_2}).$$

Next, recall again the following almost orthogonality estimate (see (2.11) in Section 2, and see also Lemma 2.11 in [HLLu]): For $\epsilon \in (0, 1)$,

$$(4.24) \quad \begin{aligned} & |\tilde{Q}_{k_1}^{(1)}(\psi_1)(x_{I_1})\tilde{Q}_{k_2}^{(2)}(\psi_2)(x_{I_2})| \\ & \lesssim \prod_{i=1}^2 2^{-|k_i - \tilde{k}_i|\epsilon} \left(\frac{2^{-k_i} + 2^{-\tilde{k}_i}}{|x_i - x_{I_i}| + 2^{-k_i} + 2^{-\tilde{k}_i}} \right)^\epsilon \\ & \quad \times \frac{1}{m_\lambda(I(x_i, 2^{-k_i} + 2^{-\tilde{k}_i})) + m_\lambda(I(x_{I_i}, 2^{-k_i} + 2^{-\tilde{k}_i})) + m_\lambda(I(x_i, |x_i - x_{I_i}|))}. \end{aligned}$$

For arbitrary dyadic intervals I_1 and I_2 , we choose $x_{I_1} \in I_1$ and $x_{I_2} \in I_2$ such that

$$\left| Q_{k_1}^{(1)}Q_{k_2}^{(2)}(g)(x_{I_1}, x_{I_2}) \right| \leq 2 \inf_{z_1 \in I_1, z_2 \in I_2} \left| Q_{k_1}^{(1)}Q_{k_2}^{(2)}(g)(z_1, z_2) \right|,$$

which implies that

$$\begin{aligned} \left| Q_{k_1}^{(1)}Q_{k_2}^{(2)}(g)(x_{I_1}, x_{I_2}) \right| & \leq 2 \inf_{z_1 \in I_1, z_2 \in I_2} \left(\left| P_{k_1}^{(1)}P_{k_2}^{(2)}(g)(z_1, z_2) \right| + \left| P_{k_1-1}^{(1)}P_{k_2}^{(2)}(g)(z_1, z_2) \right| \right. \\ & \quad \left. + \left| P_{k_1}^{(1)}P_{k_2-1}^{(2)}(g)(z_1, z_2) \right| + \left| P_{k_1-1}^{(1)}P_{k_2-1}^{(2)}(g)(z_1, z_2) \right| \right) \\ & \leq 8 \inf_{z_1 \in I_1, z_2 \in I_2} \mathcal{R}_P(f)(z_1, z_2). \end{aligned}$$

Then, based on the estimates in the proof of Theorem 2.10 in [HLLu, pp. 335–336], see also the estimates we had in Section 2.3 for \mathbb{L} in (2.10), we have the following estimate:

$$\begin{aligned} |\langle f, \psi \rangle| & \lesssim \sum_{k_1} \sum_{k_2} 2^{-|k_1 - \tilde{k}_1|\epsilon} 2^{-|k_2 - \tilde{k}_2|\epsilon} 2^{[(k_1 \wedge \tilde{k}_1) - k_1](2\lambda+1)(1-\frac{1}{r})} 2^{[(k_2 \wedge \tilde{k}_2) - k_2](2\lambda+1)(1-\frac{1}{r})} \\ & \quad \times \left[\mathcal{M}_1 \left(\sum_{I_1 \in \mathcal{X}^{k_1+N_1}} \mathcal{M}_2 \left(\sum_{I_2 \in \mathcal{X}^{k_2+N_2}} \inf_{\substack{z_1 \in I_1 \\ z_2 \in I_2}} |\mathcal{R}_P(f)(z_1, z_2)|^r \chi_{I_2}(\cdot) \right) (x_2) \chi_{I_1}(\cdot) \right) (x_1) \right]^{\frac{1}{r}} \\ & \lesssim \left[\mathcal{M}_1 \left(\mathcal{M}_2 \left(|\mathcal{R}_P(f)|^r \right) \right) (x_1, x_2) \right]^{\frac{1}{r}}, \end{aligned}$$

where $\frac{2\lambda+1}{2\lambda+2} < r < p$ and $a \wedge b := \min\{a, b\}$, which shows that (4.21) holds.

We now prove (4.20). For every $\varphi := \varphi_1 \varphi_2$ with $\|\varphi_i\|_{\mathcal{G}(x_i, t_i, \beta_i, \gamma_i)} \leq 1$, $i = 1, 2$, let

$$\sigma_1 := \int_{\mathbb{R}_+} \varphi_1(x_1) dm_\lambda(x_1), \quad \sigma_2 := \int_{\mathbb{R}_+} \varphi_2(x_2) dm_\lambda(x_2).$$

It is obvious that $|\sigma_1|, |\sigma_2| \lesssim 1$ since $\varphi_i \in \mathcal{G}(x_i, t_i, \beta_i, \gamma_i)$ for $i = 1, 2$. We set

$$\psi_1(y_1) := \frac{1}{1 + \sigma_1 \tilde{C}_\lambda} \left[\varphi(y_1) - \sigma_1 P_{\tilde{k}_1}^{(1)}(x_1, y_1) \right], \quad \psi_2(y_2) := \frac{1}{1 + \sigma_2 \tilde{C}_\lambda} \left[\varphi(y_2) - \sigma_2 P_{\tilde{k}_2}^{(2)}(x_2, y_2) \right],$$

where $\tilde{k}_i := \lfloor \log_2 t_i \rfloor + 1$ for $i = 1, 2$. Then, we see that $\psi_1 \in \mathcal{G}(x_1, 2^{-\tilde{k}_1}, \beta_1, \gamma_1)$ and $\psi_2 \in \mathcal{G}(x_2, 2^{-\tilde{k}_2}, \beta_2, \gamma_2)$. Based on the normalisation factor $\frac{1}{1 + \sigma_1 \tilde{C}_\lambda}$, we obtain that

$$\|\psi_1\|_{\mathcal{G}(x_1, 2^{-\tilde{k}_1}, \beta_1, \gamma_1)} \leq 1 \quad \text{and} \quad \|\psi_2\|_{\mathcal{G}(x_2, 2^{-\tilde{k}_2}, \beta_2, \gamma_2)} \leq 1.$$

Moreover, we point out that

$$\int_{\mathbb{R}_+} \psi_1(y_1) dm_\lambda(y_1) = 0 \quad \text{for all } x_1 \in \mathbb{R}_+$$

since $\int_{\mathbb{R}_+} P_{k_1}^{(1)}(x_1, y_1) dm_\lambda(y_1) = 1$ for all $x_1 \in \mathbb{R}_+$. Similarly we have the cancellation property for $\psi_2(y_2)$. Hence, we further obtain that $\psi_1 \in \mathring{\mathcal{G}}_1(\beta_1, \gamma_1)$ and $\psi_2 \in \mathring{\mathcal{G}}_1(\beta_2, \gamma_2)$.

Based on the definition of ψ_1 and ψ_2 , we have

$$\begin{aligned} |\langle f, \varphi \rangle| &= \left| \left\langle f, \left[\sigma_1 P_{k_1}(x_1, \cdot) + (1 + \sigma_1 \tilde{C}_\lambda) \psi_1(\cdot) \right] \left[\sigma_2 P_{k_2}(x_2, \cdot) + (1 + \sigma_2 \tilde{C}_\lambda) \psi_2(\cdot) \right] \right\rangle \right| \\ &\leq \left| \left\langle f, \sigma_1 \sigma_2 P_{k_1}(x_1, \cdot) P_{k_2}(x_2, \cdot) \right\rangle \right| + \left| \left\langle f, (1 + \sigma_1 \tilde{C}_\lambda) \psi_1(\cdot) \sigma_2 P_{k_2}(x_2, \cdot) \right\rangle \right| \\ &\quad + \left| \left\langle f, \sigma_1 P_{k_1}(x_1, \cdot) (1 + \sigma_2 \tilde{C}_\lambda) \psi_2(\cdot) \right\rangle \right| + \left| \left\langle f, (1 + \sigma_1 \tilde{C}_\lambda) \psi_1(\cdot) (1 + \sigma_2 \tilde{C}_\lambda) \psi_2(\cdot) \right\rangle \right| \\ &=: A_1 + A_2 + A_3 + A_4. \end{aligned}$$

For the term A_1 , from the definition of $\mathcal{R}_P(f)$ in Section 1, we get that

$$A_1 \lesssim \mathcal{R}_P(f)(x_1, x_2) = \left\{ |\mathcal{R}_P(f)(x_1, x_2)|^r \right\}^{\frac{1}{r}} \leq \left[\mathcal{M}_1 \left(\mathcal{M}_2 \left(|\mathcal{R}_P(f)|^r \right) \right) (x_1, x_2) \right]^{\frac{1}{r}}$$

for any $r \in (0, 1]$. For the term A_4 , from (4.21) we obtain that

$$A_4 \lesssim \left[\mathcal{M}_1 \left(\mathcal{M}_2 \left(|\mathcal{R}_P(f)|^r \right) \right) (x_1, x_2) \right]^{\frac{1}{r}}$$

for $\frac{2\lambda+1}{2\lambda+2} < r < p$. As for A_2 , let $F_{x_2}(\cdot) := \langle f, P_{k_2}(x_2, \cdot) \rangle$. Then we have

$$A_2 \sim \left| \left\langle F_{x_2}(\cdot), (1 + \sigma \tilde{C}_\lambda) \psi_1(\cdot) \right\rangle \right|.$$

Then, following the same approach above, by using the reproducing formula in terms of $Q_{k_2}^{(2)}$, the almost orthogonality estimate, we obtain that

$$A_2 \lesssim \left[\mathcal{M}_1 \left(\left[\sup_{t_1 > 0} |P_{t_1}^{[\lambda]}(F_{x_2}(\cdot))| \right]^r \right) (x_1) \right]^{\frac{1}{r}} = \left[\mathcal{M}_1 \left(\left[\sup_{t_1 > 0} |P_{t_1}^{[\lambda]} P_{2^{-k_2}}^{[\lambda]}(f)(\cdot, x_2)| \right]^r \right) (x_1) \right]^{\frac{1}{r}},$$

which is further bounded by

$$\left[\mathcal{M}_1 \left(\mathcal{M}_2 \left(|\mathcal{R}_P(f)|^r \right) \right) (x_1, x_2) \right]^{\frac{1}{r}}.$$

Similarly, we obtain that A_3 satisfies the same estimates. Combining the estimates of A_1 , A_2 , A_3 and A_4 , we obtain that (4.20) holds.

Step 4: $\|f\|_{H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\mathcal{R}_h}^p(\mathbb{R}_\lambda)}$ for $f \in H_{\mathcal{R}_h}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$.

Indeed, we recall the well-known subordination formula that for all $f \in L^2(\mathbb{R}_\lambda)$,

$$P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f(x_1, x_2) = \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{e^{-u_1}}{\sqrt{u_1}} \frac{e^{-u_2}}{\sqrt{u_2}} e^{-\frac{t_1^2}{4u_1} \Delta_\lambda} e^{-\frac{t_2^2}{4u_2} \Delta_\lambda} f(x_1, x_2) du_1 du_2.$$

From this, it follows that

$$\begin{aligned} \mathcal{R}_P f(x_1, x_2) &\lesssim \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \int_0^\infty \int_0^\infty \frac{e^{-u_1}}{\sqrt{u_1}} \frac{e^{-u_2}}{\sqrt{u_2}} \left| e^{-\frac{t_1^2}{4u_1} \Delta_\lambda} e^{-\frac{t_2^2}{4u_2} \Delta_\lambda} f(x_1, x_2) \right| du_1 du_2 \\ &\lesssim \mathcal{R}_h f(x_1, x_2) \int_0^\infty \int_0^\infty \frac{e^{-u_1}}{\sqrt{u_1}} \frac{e^{-u_2}}{\sqrt{u_2}} du_1 du_2 \\ &\lesssim \mathcal{R}_h f(x_1, x_2), \end{aligned}$$

which further implies that for all $f \in H_{\mathcal{R}_h}^p(\mathbb{R}_\lambda)$, $\|f\|_{H_{\mathcal{R}_P}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\mathcal{R}_h}^p(\mathbb{R}_\lambda)}$.

Step 5: $\|f\|_{H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda)} \leq \|f\|_{H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda)}$ for $f \in H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$.

Observe that for all $f \in L^2(\mathbb{R}_\lambda)$, $\mathcal{R}_h f \leq \mathcal{N}_h f$. Then we see that for all $f \in H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda)$,

$$\|f\|_{H_{\mathcal{R}_h}^p(\mathbb{R}_\lambda)} \leq \|f\|_{H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda)}.$$

Step 6: $\|f\|_{H_{\mathcal{N}_h}^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)}$ for $f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda) \cap L^2(\mathbb{R}_\lambda)$.

We claim that it suffices to prove that there exists a constant $\delta > 0$ such that for every H^p rectangular atom α_R as in Definition 2.7, and $\gamma_1, \gamma_2 \geq 2$,

$$(4.25) \quad \int_{x_1 \notin \gamma_1 I} \int_0^\infty |\mathcal{N}_h(\alpha_R)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \lesssim [\mu_\lambda(R)]^{1-\frac{p}{2}} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \gamma_1^{-p}$$

and

$$(4.26) \quad \int_0^\infty \int_{x_2 \notin \gamma_2 J} |\mathcal{N}_h(\alpha_R)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \lesssim [\mu_\lambda(R)]^{1-\frac{p}{2}} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \gamma_2^{-p}.$$

In fact, if (4.25) and (4.26) hold, then we can obtain that for every H^p atom a , we have

$$(4.27) \quad \int_0^\infty \int_0^\infty |\mathcal{N}_h(a)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \lesssim 1.$$

To see this, suppose a is supported in an open set $\Omega \subset \mathbb{R}_+ \times \mathbb{R}_+$ with finite measure and $a := \sum_{R \in m(\Omega)} \alpha_R$. We now define

$$\tilde{\Omega} := \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{M}_S(\chi_\Omega)(x_1, x_2) > 1/2\}$$

$$\tilde{\tilde{\Omega}} := \{(x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+ : \mathcal{M}_S(\chi_{\tilde{\Omega}})(x_1, x_2) > 1/2\}.$$

Moreover, for every $R := I \times J \in m_1(\Omega)$ (see Section 2.1 for the definition of $m_1(\Omega)$), let \tilde{I} be the largest dyadic interval containing I such that $\tilde{R} := \tilde{I} \times J \subset \tilde{\Omega}$ and let \tilde{J} be the largest dyadic interval containing J such that $\tilde{\tilde{R}} := \tilde{I} \times \tilde{J} \subset \tilde{\tilde{\Omega}}$. We now let $\gamma_1 := \frac{|\tilde{I}|}{|I|}$ and $\gamma_2 := \frac{|\tilde{J}|}{|J|}$.

Then we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty |\mathcal{N}_h(a)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \\ &= \left(\iint_{\cup_{R \in m(\Omega)} 10\tilde{R}} + \iint_{(\cup_{R \in m(\Omega)} 10\tilde{R})^c} \right) |\mathcal{N}_h(a)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \\ &:= A_1 + A_2. \end{aligned}$$

For the term A_1 , using Hölder's inequality and the $L^2(\mathbb{R}_\lambda)$ -boundedness of \mathcal{N}_h , we have that

$$\begin{aligned} A_1 &\leq \left[\mu_\lambda \left(\bigcup_{R \in m(\Omega)} 10\tilde{R} \right) \right]^{1-\frac{p}{2}} \left(\iint_{\cup_{R \in m(\Omega)} 10\tilde{R}} |\mathcal{N}_h(a)(x_1, x_2)|^2 d\mu_\lambda(x_1, x_2) \right)^{\frac{p}{2}} \\ &\lesssim \left[\mu_\lambda(\Omega) \right]^{1-\frac{p}{2}} \|a\|_{L^2(\mathbb{R}_\lambda)}^p \lesssim \left[\mu_\lambda(\Omega) \right]^{1-\frac{p}{2}} \left[\mu_\lambda(\Omega)^{\frac{1}{2}-\frac{1}{p}} \right]^p \lesssim 1. \end{aligned}$$

For the term A_2 , we have

$$\begin{aligned} A_2 &\leq \sum_{R \in m(\Omega)} \iint_{(10\tilde{R})^c} |\mathcal{N}_h(\alpha_R)(x_1, x_2)|^2 d\mu_\lambda(x_1, x_2) \\ &\leq \sum_{R \in m(\Omega)} \iint_{(10\tilde{I})^c \times \mathbb{R}_+} |\mathcal{N}_h(\alpha_R)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \end{aligned}$$

$$\begin{aligned}
 & + \sum_{R \in m(\Omega)} \iint_{\mathbb{R}_+ \times (10\tilde{I})^c} |\mathcal{N}_h(\alpha_R)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \\
 & =: A_{21} + A_{22}.
 \end{aligned}$$

From (4.25) and (4.26), we have

$$A_{21} \lesssim \sum_{R \in m(\Omega)} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \mu_\lambda(R)^{1-\frac{p}{2}} \left(\frac{|I|}{|\tilde{I}|} \right)^{-p}$$

and that

$$A_{22} \lesssim \sum_{R \in m(\Omega)} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \mu_\lambda(R)^{1-\frac{p}{2}} \left(\frac{|J|}{|\tilde{J}|} \right)^{-p}.$$

As a consequence, using Hölder's inequality and Journé's covering lemma ([J1], [P], see also the version on spaces of homogeneous type in [HLLin]) we get that

$$\begin{aligned}
 A_2 & \lesssim \left(\sum_{R \in m(\Omega)} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^2 \right)^{\frac{p}{2}} \left(\sum_{R \in m(\Omega)} \mu_\lambda(R) \left(\frac{|I|}{|\tilde{I}|} \right)^{-2p} \right)^{1-\frac{p}{2}} \\
 & + \left(\sum_{R \in m(\Omega)} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^2 \right)^{\frac{p}{2}} \left(\sum_{R \in m(\Omega)} \mu_\lambda(R) \left(\frac{|J|}{|\tilde{J}|} \right)^{-2p} \right)^{1-\frac{p}{2}} \\
 & \lesssim \mu_\lambda(\Omega)^{\frac{p}{2}-1} \mu_\lambda(\Omega)^{1-\frac{p}{2}} \lesssim 1.
 \end{aligned}$$

Combining the estimates of the two terms A_1 and A_2 , we get that (4.27) holds.

Now, based on (4.27), for every $f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, we have that $f = \sum_j \lambda_j a_j$ with $\sum_j |\lambda_j|^p \sim \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)}^p$. Hence,

$$\begin{aligned}
 \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\mathcal{N}_h(f)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) & \leq \sum_j |\lambda_j|^p \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |\mathcal{N}_h(a_j)(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \\
 & \lesssim \sum_j |\lambda_j|^p \lesssim \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)}^p.
 \end{aligned}$$

Thus, to prove **Step 6**, it suffices to prove (4.25) and (4.26). By symmetry, we only prove (4.25). To this end, we write

$$\begin{aligned}
 & \int_{x_1 \notin \gamma I} \int_0^\infty |\mathcal{N}_h \alpha_R(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \\
 & = \left[\sum_{k=0}^\infty \int_{2^{k+1}\gamma I \setminus 2^k\gamma I} \int_{8J} + \sum_{k=0}^\infty \int_{2^{k+1}\gamma I \setminus 2^k\gamma I} \int_{\mathbb{R}_+ \setminus 8J} \right] |\mathcal{N}_h \alpha_R(x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \\
 & =: F_1 + F_2.
 \end{aligned}$$

For F_1 , since

$$\mathcal{N}_h \alpha_R(x_1, x_2) \leq \sup_{|x_2 - y_2| < t_2} W_{t_2}^{[\lambda]} \left(\sup_{|x_1 - y_1| < t_1} |W_{t_1}^{[\lambda]} \alpha_R(y_1, \cdot)| \right)(y_2)$$

by Hölder's inequality and the $L^2(\mathbb{R}_+, dm_\lambda)$ -boundedness of $\sup_{|x_2 - y_2| < t_2} |W_{t_2}^{[\lambda]} f(y)|$, we have that

$$F_1 \leq \sum_{k=0}^\infty [m_\lambda(8J)]^{1-\frac{p}{2}} \int_{2^{k+1}\gamma I \setminus 2^k\gamma I} \left[\int_{8J} |\mathcal{N}_h \alpha_R(x_1, x_2)|^2 dm_\lambda(x_2) \right]^{\frac{p}{2}} dm_\lambda(x_1)$$

$$\lesssim [m_\lambda(J)]^{1-\frac{p}{2}} \sum_{k=0}^{\infty} \int_{2^{k+1}\gamma I \setminus 2^k\gamma I} \left[\int_J \left[\sup_{|x_1-y_1|<t_1} |W_{t_1}^{[\lambda]} \alpha_R(\cdot, x_2)(x_1)| \right]^2 dm_\lambda(x_2) \right]^{\frac{p}{2}} dm_\lambda(x_1).$$

Since for any fixed x_2 , $\int_0^\infty \alpha_R(z_1, x_2) dm_\lambda(z_1) = 0$, we conclude that

$$\begin{aligned} |W_{t_1}^{[\lambda]} \alpha_R(\cdot, x_2)(x_1)| &= \left| \int_I [W_{t_1}^{[\lambda]}(y_1, z_1) - W_{t_1}^{[\lambda]}(y_1, x_0^1)] \alpha_R(z_1, x_2) dm_\lambda(z_1) \right| \\ &\lesssim \int_I \frac{t_1 |z_1 - x_0^1|}{m_\lambda(I(x_0^1, |x_1 - x_0^1|))(|x_0^1 - y_1| + t_1)^2} |\alpha_R(z_1, x_2)| dm_\lambda(z_1), \end{aligned}$$

where x_0^1 is the center of I , and the last inequality follows from the fact that $W_{t_1}^{[\lambda]}(y_1, z_1)$ as a function of z_1 satisfies (K_{ii}) in Section 2.3 (see [YY]). Thus, observing that $|x_1 - x_0^1| \leq |x_0^1 - y_1| + t_1$, we have

$$\sup_{|x_1-y_1|<t_1} |W_{t_1}^{[\lambda]} \alpha_R(\cdot, x_2)(x_1)| \lesssim \int_I \frac{|I|}{m_\lambda(I(x_0^1, |x_1 - x_0^1|))|x_1 - x_0^1|} |\alpha_R(z_1, x_2)| dm_\lambda(z_1),$$

As a consequence, we obtain that

$$\begin{aligned} F_1 &\lesssim [m_\lambda(J)]^{1-\frac{p}{2}} [m_\lambda(I)]^{\frac{p}{2}} \sum_{k=0}^{\infty} \int_{2^{k+1}\gamma I \setminus 2^k\gamma I} \frac{|I|^p}{|x_1 - x_0^1|^p} \frac{1}{m_\lambda(I(x_0^1, |x_1 - x_0^1|))^p} \\ &\quad \times \left[\int_J \int_I [\alpha_R(z_1, x_2)]^2 dm_\lambda(z_1) dm_\lambda(x_2) \right]^{\frac{p}{2}} dm_\lambda(x_1) \\ &\lesssim [m_\lambda(J)]^{1-\frac{p}{2}} [m_\lambda(I)]^{\frac{p}{2}} \gamma^{-p} \sum_{k=0}^{\infty} \frac{m_\lambda(2^{k+1}I)^{1-p}}{2^{kp}} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \\ &\lesssim [m_\lambda(J)]^{1-\frac{p}{2}} [m_\lambda(I)]^{\frac{p}{2}} \gamma^{-p} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p m_\lambda(I)^{1-p} \sum_{k=0}^{\infty} \frac{2^{(k+1)(2\lambda+1)(1-p)}}{2^{kp}} \\ &\lesssim [\mu_\lambda(R)]^{1-\frac{p}{2}} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \gamma^{-p}, \end{aligned}$$

where the last inequality follows from the condition that $p \in (\frac{2\lambda+1}{2\lambda+2}, 1]$.

For F_2 , let x_0^2 be the center of J . By the cancellation of α_R and the property (K_{ii}) for $W_{t_1}^{[\lambda]}(y_1, z_1)$ and $W_{t_2}^{[\lambda]}(y_2, z_2)$, we also have

$$\begin{aligned} &\mathcal{N}_h \alpha_R(x_1, x_2) \\ &\leq \sup_{\substack{|y_1-x_1|<t_1 \\ |y_2-x_2|<t_2}} \int_I \int_J \left| W_{t_1}^{[\lambda]}(y_1, z_1) - W_{t_1}^{[\lambda]}(y_1, x_0^1) \right| \left| W_{t_2}^{[\lambda]}(y_2, z_2) - W_{t_2}^{[\lambda]}(y_2, x_0^2) \right| |\alpha_R(z_1, z_2)| d\mu_\lambda(z_1, z_2) \\ &\lesssim \sup_{\substack{|y_1-x_1|<t_1 \\ |y_2-x_2|<t_2}} \int_I \int_J \frac{1}{m_\lambda(I(x_0^1, |x_0^1 - x_1|))} \frac{t_1 |I|}{(|x_0^1 - y_1| + t_1)^2} \\ &\quad \times \frac{1}{m_\lambda(I(x_0^2, |x_0^2 - x_2|))} \frac{t_2 |J|}{(|x_0^2 - y_2| + t_2)^2} |\alpha_R(z_1, z_2)| d\mu_\lambda(z_1, z_2) \\ &\lesssim [\mu_\lambda(R)]^{\frac{1}{2}} \frac{|I|}{|x_1 - x_0^1|} \frac{1}{m_\lambda(I(x_0^1, |x_1 - x_0^1|))} \frac{|J|}{|x_2 - x_0^2|} \frac{1}{m_\lambda(I(x_0^2, |x_2 - x_0^2|))} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)} \\ &\lesssim \frac{|I|}{|x_1 - x_0^1|} \frac{1}{m_\lambda(I(x_0^1, |x_1 - x_0^1|))} \frac{|J|}{|x_2 - x_0^2|} \frac{1}{m_\lambda(I(x_0^2, |x_2 - x_0^2|))} [\mu_\lambda(R)]^{\frac{1}{2}} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}. \end{aligned}$$

Therefore,

$$\begin{aligned} F_2 &\lesssim [\mu_\lambda(R)]^{\frac{p}{2}} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \sum_{k=0}^{\infty} \sum_{l=3}^{\infty} \int_{2^{k+1}\gamma I \setminus 2^k\gamma I} \int_{2^{l+1}J \setminus 2^lJ} \frac{|I|^p}{|x_1 - x_0^1|^p} \frac{1}{m_\lambda(I(x_0^1, |x_1 - x_0^1|))^p} \\ &\quad \times \frac{|J|^p}{|x_2 - x_0^2|^p} \frac{1}{m_\lambda(I(x_0^2, |x_2 - x_0^2|))^p} dm_\lambda(x_2) dm_\lambda(x_1) \\ &\lesssim [\mu_\lambda(R)]^{1-\frac{p}{2}} \|\alpha_R\|_{L^2(\mathbb{R}_\lambda)}^p \gamma^{-p}. \end{aligned}$$

Combining the estimates of F_1 and F_2 , we obtain (4.25). We finish the proof of Theorem 1.6.

5. PROOFS OF THEOREMS 1.8 AND 1.9

In this section, we present the proofs of Theorems 1.8 and 1.9. We first note that $R_{\Delta_\lambda,1}R_{\Delta_\lambda,2}$ is a product Calderón–Zygmund operator on space of homogeneous type \mathbb{R}_λ (see the definition in Section 1 of [HLLin]). And we consider $R_{\Delta_\lambda,1}$ as $R_{\Delta_\lambda,1} \otimes Id_2$ and $R_{\Delta_\lambda,2}$ as $Id_1 \otimes R_{\Delta_\lambda,2}$, where we use Id_1 and Id_2 to denote the identity operator on $L^2(\mathbb{R}_+, dm_\lambda)$. Then we can also understand $R_{\Delta_\lambda,1}$ and $R_{\Delta_\lambda,2}$ as product Calderón–Zygmund operators on \mathbb{R}_λ . We recall that the product Calderón–Zygmund operators T are bounded on $L^r(\mathbb{R}_\lambda)$ for $r \in (1, \infty)$, on $H^p(\mathbb{R}_\lambda)$ ([HLLin, Section 3.1]) and from $H^p(\mathbb{R}_\lambda)$ to $L^p(\mathbb{R}_\lambda)$ for $p \in ((2\lambda+1)/(2\lambda+2), 1]$ ([HLLin, Section 2.1.3]). Hence, for any $f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, we have

$$(5.1) \quad \|R_{\Delta_\lambda,1}(f)\|_{L^p(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda,2}(f)\|_{L^p(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda,1}R_{\Delta_\lambda,2}(f)\|_{L^p(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)}.$$

Before we give the proof of Theorem 1.8, we first recall the following result from Lemma 11 in [MSt].

Lemma 5.1. *Suppose that*

- (i) $u(t, x)$ is continuous in $t \in [0, \infty)$, $x \in \mathbb{R}$ and even in x ;
- (ii) In the region where $u(t, x) > 0$, u is of class C^2 and satisfies $\partial_t^2 u + \partial_x^2 u + 2\lambda x^{-1} \partial_x u \geq 0$;
- (iii) $u(0, x) = 0$;
- (iv) For some $r \in [1, \infty)$, there exists a positive constant \tilde{C} such that

$$\sup_{0 < t < \infty} \int_0^\infty |u(t, x)|^r dm_\lambda(x) \leq \tilde{C} < \infty.$$

Then $u(t, x) \leq 0$.

For $f \in L^p(\mathbb{R}_\lambda)$ with $p \in [1, \infty)$, and $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$, let

$$(5.2) \quad u(t_1, t_2, x_1, x_2) := P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f(x_1, x_2), \quad v(t_1, t_2, x_1, x_2) := Q_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f(x_1, x_2),$$

and

$$(5.3) \quad w(t_1, t_2, x_1, x_2) := P_{t_1}^{[\lambda]} Q_{t_2}^{[\lambda]} f(x_1, x_2), \quad z(t_1, t_2, x_1, x_2) := Q_{t_1}^{[\lambda]} Q_{t_2}^{[\lambda]} f(x_1, x_2),$$

where $Q_{t_1}^{[\lambda]}$ and $Q_{t_2}^{[\lambda]}$ are defined as in (2.3). Moreover, define

$$(5.4) \quad u^*(x_1, x_2) := \mathcal{R}_P f(x_1, x_2)$$

and

$$(5.5) \quad F(t_1, t_2, x_1, x_2) := \{[u(t_1, t_2, x_1, x_2)]^2 + [v(t_1, t_2, x_1, x_2)]^2 + [w(t_1, t_2, x_1, x_2)]^2 + [z(t_1, t_2, x_1, x_2)]^2\}^{\frac{1}{2}}.$$

We first establish the following lemma.

Lemma 5.2. *Let $f \in H_{Riesz}^1(\mathbb{R}_\lambda)$, u, v, w, z and F be, respectively, as in (5.2), (5.3) and (5.5). Then there exists a positive constant C independent of f, u, v, w, z and F , such that*

$$\sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F(t_1, t_2, x_1, x_2) d\mu_\lambda(x_1, x_2) \leq C \|f\|_{H_{Riesz}^1(\mathbb{R}_\lambda)}.$$

Proof. It suffices to show that

$$(5.6) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |u(t_1, t_2, x_1, x_2)| d\mu_\lambda(x_1, x_2) \leq \|f\|_{L^1(\mathbb{R}_\lambda)},$$

$$(5.7) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |v(t_1, t_2, x_1, x_2)| d\mu_\lambda(x_1, x_2) \lesssim \|R_{\Delta_\lambda, 1} f\|_{L^1(\mathbb{R}_\lambda)},$$

$$(5.8) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |w(t_1, t_2, x_1, x_2)| d\mu_\lambda(x_1, x_2) \lesssim \|R_{\Delta_\lambda, 2} f\|_{L^1(\mathbb{R}_\lambda)},$$

and

$$(5.9) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |z(t_1, t_2, x_1, x_2)| d\mu_\lambda(x_1, x_2) \lesssim \|R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2} f\|_{L^1(\mathbb{R}_\lambda)}.$$

To this end, we first note that (5.6) follows from (S_i) in Lemma 2.9. Moreover, by the fact that for any $t, y \in \mathbb{R}_+$,

$$(5.10) \quad \int_0^\infty xy P_t^{[\lambda+1]}(x, y) dm_\lambda(x) \lesssim 1,$$

we obtain that for every $f \in L^1(\mathbb{R}_+, dm_\lambda)$ and $t \in \mathbb{R}_+$,

$$(5.11) \quad \begin{aligned} \|Q_t^{[\lambda]} f\|_{L^1(\mathbb{R}_+, dm_\lambda)} &= \left\| \int_0^\infty \cdot y P_t^{[\lambda+1]}(\cdot, y) R_{\Delta_\lambda}(f)(y) dm_\lambda(y) \right\|_{L^1(\mathbb{R}_+, dm_\lambda)} \\ &\lesssim \|R_{\Delta_\lambda} f\|_{L^1(\mathbb{R}_+, dm_\lambda)}; \end{aligned}$$

see [BDT, p. 208]. Therefore, by the uniform $L^1(\mathbb{R}_+, dm_\lambda)$ -boundedness of $\{P_t^{[\lambda]}\}_{t>0}$ (Lemma 2.9 (S_i)), we see that for any $t_1, t_2 \in \mathbb{R}_+$,

$$\iint_{\mathbb{R}_+ \times \mathbb{R}_+} |v(t_1, t_2, x_1, x_2)| d\mu_\lambda(x_1, x_2) \leq \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left| Q_{t_1}^{[\lambda]} f(x_1, x_2) \right| d\mu_\lambda(x_1, x_2) \lesssim \|R_{\Delta_\lambda, 1} f\|_{L^1(\mathbb{R}_\lambda)}.$$

This implies (5.7). Similarly, we have (5.8).

Finally, from (5.11), we deduce that

$$z(t_1, t_2, x_1, x_2) = \int_0^\infty \int_0^\infty x_1 y_1 P_{t_1}^{[\lambda+1]}(x_1, y_1) x_2 y_2 P_{t_2}^{[\lambda+1]}(x_2, y_2) R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2} f(y_1, y_2) d\mu_\lambda(y_1, y_2).$$

By this and (5.10), we show (5.9) immediately. This finishes the proof of Lemma 5.2. \square

Proof of Theorem 1.8. We first show that for any $f \in H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)$,

$$\|f\|_{H_{Riesz}^1(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)}.$$

To see this, based on Definition 1.7 and Theorem 1.2, it suffices to prove that

$$\|f\|_{L^1(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda, 1} f\|_{L^1(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda, 2} f\|_{L^1(\mathbb{R}_\lambda)} + \|R_{\Delta_\lambda, 1} R_{\Delta_\lambda, 2} f\|_{L^1(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)},$$

which follows from (5.1) with $p := 1$ and the fact that $H_{\Delta_\lambda}^1(\mathbb{R}_\lambda)$ is a subspace of $L^1(\mathbb{R}_\lambda)$.

Conversely, assume that $f \in H_{Riesz}^1(\mathbb{R}_\lambda)$. By Theorem 1.6, it suffices to show that

$$\|f\|_{H_{\mathcal{R}_P}^1(\mathbb{R}_\lambda)} \lesssim \|f\|_{H_{Riesz}^1(\mathbb{R}_\lambda)}.$$

To this end, based on Lemma 5.2, it remains to prove that

$$(5.12) \quad \|f\|_{H_{\mathcal{R}_P}^1(\mathbb{R}_\lambda)} = \|u^*\|_{L^1(\mathbb{R}_\lambda)} \lesssim \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F(t_1, t_2, x_1, x_2) d\mu_\lambda(x_1) d\mu_\lambda(x_2),$$

where u^* and F are as in (5.4) and (5.5). We first claim that we only need to show that for $p \in (\frac{2\lambda+1}{2\lambda+2}, 1)$ and $\epsilon_1, t_1, \epsilon_2, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$(5.13) \quad F^p(\epsilon_1 + t_1, \epsilon_2 + t_2, x_1, x_2) \lesssim P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} (F^p(\epsilon_1, \epsilon_2, \cdot, \cdot))(x_1, x_2).$$

Indeed, by Lemma 5.2, we see that $F \in L^1(\mathbb{R}_\lambda)$. If (5.13) holds, then the uniform $L^r(\mathbb{R}_+, d\mu_\lambda)$ -boundedness of $\{P_t^{[\lambda]}\}_{t>0}$, with $r := 1/p$, implies that $\{F^p(\epsilon_1, \epsilon_2, \cdot, \cdot)\}_{\epsilon_1, \epsilon_2 > 0}$ is bounded in $L^r(\mathbb{R}_\lambda)$. Since $L^r(\mathbb{R}_\lambda)$ is reflexive, there exist two sequences $\{\epsilon_{1,k}\}, \{\epsilon_{2,j}\} \downarrow 0$ and $h \in L^r(\mathbb{R}_\lambda)$ such that $\{F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot)\}_{\epsilon_{1,k}, \epsilon_{2,j} > 0}$ converges weakly to h in $L^r(\mathbb{R}_\lambda)$ as $k, j \rightarrow \infty$. Moreover, by Hölder's inequality, we see that

$$(5.14) \quad \begin{aligned} \|h\|_{L^r(\mathbb{R}_\lambda)}^r &= \left\{ \sup_{\|g\|_{L^{r'}(\mathbb{R}_\lambda)} \leq 1} \left| \iint_{\mathbb{R}_+ \times \mathbb{R}_+} g(x_1, x_2) h(x_1, x_2) d\mu_\lambda(x_1, x_2) \right| \right\}^r \\ &= \left\{ \sup_{\|g\|_{L^{r'}(\mathbb{R}_\lambda)} \leq 1} \lim_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} \left| \iint_{\mathbb{R}_+ \times \mathbb{R}_+} g(x_1, x_2) F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot) d\mu_\lambda(x_1, x_2) \right| \right\}^r \\ &\leq \limsup_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} \|F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot)\|_{L^r(\mathbb{R}_\lambda)}^r \\ &\leq \sup_{t_1 > 0, t_2 > 0} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F(t_1, t_2, x_1, x_2) d\mu_\lambda(x_1, x_2). \end{aligned}$$

Since F is continuous in t_1 and t_2 , for any $x_1, x_2 \in \mathbb{R}_+$,

$$F^p(t_1 + \epsilon_{1,k}, t_2 + \epsilon_{2,j}, x_1, x_2) \rightarrow F^p(t_1, t_2, x_1, x_2)$$

as $k, j \rightarrow \infty$. Observe that for each $x_1, x_2 \in (0, \infty)$,

$$P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} (F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot))(x_1, x_2) \rightarrow P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} (h)(x_1, x_2)$$

as $k, j \rightarrow \infty$. Thus, by these facts and (5.13), we have that for any $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$\begin{aligned} F^p(t_1, t_2, x_1, x_2) &= \lim_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} F^p(t_1 + \epsilon_{1,k}, t_2 + \epsilon_{2,j}, x_1, x_2) \\ &\lesssim \lim_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} (F^p(\epsilon_{1,k}, \epsilon_{2,j}, \cdot, \cdot))(x_1, x_2) \\ &= P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} (h)(x_1, x_2). \end{aligned}$$

Therefore,

$$[u^*(x_1, x_2)]^p \leq \sup_{t_1 > 0, t_2 > 0} F^p(t_1, t_2, x_1, x_2) \lesssim \mathcal{R}_P(h)(x_1, x_2).$$

By this together with $r := 1/p$, the $L^r(\mathbb{R}_\lambda)$ -boundedness of \mathcal{R}_P and (5.14), we then have

$$\|u^*\|_{L^1(\mathbb{R}_\lambda)} \lesssim \left\| \mathcal{R}_P(h) \right\|_{L^r(\mathbb{R}_\lambda)}^r \lesssim \sup_{t_1 > 0, t_2 > 0} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} F(t_1, t_2, x_1, x_2) d\mu_\lambda(x_1, x_2),$$

which implies that (5.12). Thus the claim holds.

Now we prove (5.13). Observe that for any fixed $t_2, x_2 \in \mathbb{R}_+$, u, v and w, z respectively satisfy the Cauchy-Riemann equations for t_1 and x_1 , and for any fixed $t_1, x_1 \in \mathbb{R}_+$, u, w and v, z respectively satisfy the Cauchy-Riemann equations for t_2 and x_2 . That is,

$$(5.15) \quad \begin{cases} \partial_{x_1} u + \partial_{t_1} v = 0, \\ \partial_{t_1} u - \partial_{x_1} v = \frac{2\lambda}{x_1} v; \end{cases} \quad \begin{cases} \partial_{x_1} w + \partial_{t_1} z = 0, \\ \partial_{t_1} w - \partial_{x_1} z = \frac{2\lambda}{x_1} z; \end{cases}$$

and

$$(5.16) \quad \begin{cases} \partial_{x_2} u + \partial_{t_2} w = 0, \\ \partial_{t_2} u - \partial_{x_2} w = \frac{2\lambda}{x_2} w; \end{cases} \quad \begin{cases} \partial_{x_2} v + \partial_{t_2} z = 0, \\ \partial_{t_2} v - \partial_{x_2} z = \frac{2\lambda}{x_2} z. \end{cases}$$

For fixed $t_2, x_2 \in \mathbb{R}_+$, let

$$F_1(t_1, t_2, x_1, x_2) := \{[u(t_1, t_2, x_1, x_2)]^2 + [v(t_1, t_2, x_1, x_2)]^2\}^{\frac{1}{2}},$$

where $t_1, x_1 \in \mathbb{R}_+$. For the moment, we fix t_2, x_2 and regard F_1 as a function of t_1 and x_1 . By (5.15), for $p \in (\frac{2\lambda+1}{2\lambda+2}, 1)$,

$$(5.17) \quad \partial_{t_1}^2 F_1^p(t_1, t_2, x_1, x_2) + \partial_{x_1}^2 F_1^p(t_1, t_2, x_1, x_2) + \frac{2\lambda}{x_1} \partial_{x_1} F_1^p(t_1, t_2, x_1, x_2) \geq 0;$$

see [MSt, Lemma 5] or [BDT, p.206]. We also observe that for any $\epsilon_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$(5.18) \quad \lim_{t_1 \rightarrow 0} P_{t_1}^{[\lambda]}(F_1^p(\epsilon_1, t_2, \cdot, x_2))(x_1) = F_1^p(\epsilon_1, t_2, x_1, x_2),$$

and by Lemma 5.2, for all $t_2 \in \mathbb{R}_+$ and almost $x_2 \in \mathbb{R}_+$,

$$(5.19) \quad \sup_{t_1 > 0} \int_0^\infty [F_1^p(t_1, t_2, x_1, x_2)]^r dm_\lambda(x_1) \leq \sup_{t_1 > 0} \int_0^\infty F(t_1, t_2, x_1, x_2) dm_\lambda(x_1) < \infty.$$

Now we claim that for any $\epsilon_1, t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$(5.20) \quad F_1^p(\epsilon_1 + t_1, t_2, x_1, x_2) \leq P_{t_1}^{[\lambda]}(F_1^p(\epsilon_1, t_2, \cdot, x_2))(x_1).$$

Indeed, as in [BDT], for any $\epsilon_1, t_1, t_2, x_1, x_2 \in \mathbb{R}_+$, let

$$\tilde{F}_{1,t_2,x_2}(t_1, x_1) := \{[\tilde{u}(t_1, t_2, x_1, x_2)]^2 + [\tilde{v}(t_1, t_2, x_1, x_2)]^2\}^{\frac{1}{2}}$$

and

$$U_{\epsilon_1, t_2, x_2}(t_1, x_1) := \tilde{F}_{1,t_2,x_2}^p(\epsilon_1 + t_1, x_1) - \widetilde{P_{t_1}^{[\lambda]}[\tilde{F}_{1,t_2,x_2}^p(\epsilon_1, \cdot)]}(x_1),$$

where for fixed t_2 and x_2 , $\widetilde{P_{t_1}^{[\lambda]}(\tilde{F}_{1,t_2,x_2}^p)}$, \tilde{u} are even extensions of $P_{t_1}^{[\lambda]}(\tilde{F}_{1,t_2,x_2}^p)$ and u , and \tilde{v} is the odd extension of v with respect to x_1 to $\mathbb{R}_+ \times \mathbb{R}$, respectively. By (1.1), (5.17), (5.18) and (5.19), it is not difficult to check that U_{ϵ_1, t_2, x_2} satisfies (i)-(iv) of Lemma 5.1. Then an application of Lemma 5.1 shows that $U_{\epsilon_1, t_2, x_2}(t_1, x_1) \leq 0$. Thus, (5.20) holds.

Similarly, let

$$F_2(t_1, t_2, x_1, x_2) := \{[w(t_1, t_2, x_1, x_2)]^2 + [z(t_1, t_2, x_1, x_2)]^2\}^{\frac{1}{2}}.$$

Since (5.17), (5.18) and (5.19) all hold with F_1 replaced with F_2 , we have that for any $\epsilon_1, t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$(5.21) \quad F_2^p(\epsilon_1 + t_1, t_2, x_1, x_2) \leq P_{t_1}^{[\lambda]}(F_2^p(\epsilon_1, t_2, \cdot, x_2))(x_1).$$

Observe that for any $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$F(t_1, t_2, x_1, x_2) \sim \sum_{i=1}^2 F_i(t_1, t_2, x_1, x_2).$$

By this fact, (5.20), (5.21) and Lemma 2.9 (S_{ii}), we have that

$$(5.22) \quad F^p(\epsilon_1 + t_1, t_2, x_1, x_2) \lesssim P_{t_1}^{[\lambda]}(F^p(\epsilon_1, t_2, \cdot, \cdot))(x_1, x_2).$$

Moreover, from (5.16), Lemma 5.2 and Lemma 5.1, we also deduce that

$$F^p(t_1, \epsilon_2 + t_2, x_1, x_2) \lesssim P_{t_2}^{[\lambda]} (F^p(t_1, \epsilon_2, \cdot, \cdot)) (x_1, x_2).$$

Now by this and (5.22), we conclude that

$$\begin{aligned} F^p(\epsilon_1 + t_1, \epsilon_2 + t_2, x_1, x_2) &\lesssim P_{t_1}^{[\lambda]} (F^p(\epsilon_1, \epsilon_2 + t_2, \cdot, \cdot)) (x_1, x_2) \\ &\lesssim P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} (F^p(\epsilon_1, \epsilon_2, \cdot, \cdot)) (x_1, x_2). \end{aligned}$$

This implies (5.13), and hence finishes the proof of Theorem 1.8. \square

We next present the proof of Theorem 1.9.

Proof of Theorem 1.9. We first assume that $f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$. By Theorem 1.6, we see that for any $f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$, $P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \in L^p(\mathbb{R}_\lambda)$ with

$$\left\| P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \right\|_{L^p(\mathbb{R}_\lambda)} \leq \| \mathcal{R}_P f \|_{L^p(\mathbb{R}_\lambda)} \lesssim \| f \|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)},$$

where the implicit constant is independent of t_1, t_2 and f . Moreover, by the semigroup property of $\{P_t^{[\lambda]}\}_{t>0}$ (Lemma 2.9) and Theorem 1.6, we obtain that

$$\left\| P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \right\|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)} \sim \left\| \mathcal{R}_P \left(P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \right) \right\|_{L^p(\mathbb{R}_\lambda)} = \| \mathcal{R}_P(f) \|_{L^p(\mathbb{R}_\lambda)} \lesssim \| f \|_{H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)}.$$

This implies $P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$ with the norm independent of t_1 and t_2 . Then an application of (5.1) shows that (1.6) holds.

Conversely, we assume that f satisfies (1.6) and show $f \in H_{\Delta_\lambda}^p(\mathbb{R}_\lambda)$. The proof is similar to that of Theorem 1.8. Precisely, we first claim that

$$(5.23) \quad \sup_{\substack{t_1 > 0 \\ t_2 > 0}} \iint_{\mathbb{R}_+ \times \mathbb{R}_+} |F(t_1, t_2, x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \lesssim 1,$$

where $F(t_1, t_2, x_1, x_2)$ is as in (5.5). From Proposition 2.11, we see that the definition of F makes sense. Indeed, for $\delta_1, \delta_2, t_1, t_2, x_1, x_2 \in \mathbb{R}_+$, let

$$u(\delta_1, \delta_2, t_1, t_2, x_1, x_2) := P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2),$$

$$v(\delta_1, \delta_2, t_1, t_2, x_1, x_2) := Q_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2),$$

$$w(\delta_1, \delta_2, t_1, t_2, x_1, x_2) := P_{t_1}^{[\lambda]} Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2),$$

and

$$z(\delta_1, \delta_2, t_1, t_2, x_1, x_2) := Q_{t_1}^{[\lambda]} Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2).$$

Moreover, as functions of variables $(\delta_1, \delta_2, t_1, t_2, x_1, x_2)$, we define

$$G_1 := \{u^2 + v^2\}^{\frac{1}{2}}, \quad G_2 := \{w^2 + z^2\}^{\frac{1}{2}}, \quad G_3 := \{u^2 + w^2\}^{\frac{1}{2}}, \quad G_4 := \{v^2 + z^2\}^{\frac{1}{2}}$$

and

$$G := \{u^2 + v^2 + w^2 + z^2\}^{\frac{1}{2}}.$$

We now prove that for all $\delta_1, \delta_2, t_1, t_2, x_1, x_2 \in \mathbb{R}_+$, when $i := 1, 2$,

$$(5.24) \quad G_i^p(\delta_1, \delta_2, t_1, t_2, x_1, x_2) \leq P_{t_1}^{[\lambda]} (G_i^p(\delta_1, \delta_2, 0, t_2, \cdot, x_2)) (x_1),$$

and when $i := 3, 4$,

$$(5.25) \quad G_i^p(\delta_1, \delta_2, t_1, t_2, x_1, x_2) \leq P_{t_2}^{[\lambda]} (G_i^p(\delta_1, \delta_2, t_1, 0, x_1, \cdot)) (x_2).$$

Here we mention that for any function $g \in L^r(\mathbb{R}_+, dm_\lambda)$ with $r \in [1, \infty)$,

$$P_0^{[\lambda]} g := \lim_{t \rightarrow 0} P_t^{[\lambda]} g = g \quad \text{and} \quad Q_0^{[\lambda]} g := \lim_{t \rightarrow 0} Q_t g = R_{\Delta_\lambda} g;$$

see [BDT, BFBMT]. Since the assumption that f is restricted at infinity implies that $P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \in L^s(\mathbb{R}_\lambda)$ for all $t_1, t_2 \in \mathbb{R}_+$ and $s \in [p, \infty]$, $G_i(\delta_1, \delta_2, t_1, 0, x_1, x_2)$ and $G_i(\delta_1, \delta_2, 0, t_2, x_1, x_2)$ make sense for $i \in \{1, 2, 3, 4\}$.

By similarity, we only show that (5.24) holds for G_2 . We fix $\delta_1, \delta_2, t_2, x_2 \in \mathbb{R}_+$ and regard G_2 as a function of t_1 and x_1 for the moment and the argument is analogous to that for (5.20). Indeed, let

$$V_{\delta_1, \delta_2, t_2, x_2}(t_1, x_1) := \tilde{G}_{2, \delta_1, \delta_2, t_2, x_2}^p(t_1, x_1) - \widetilde{P_{t_1}^{[\lambda]}} \left(\tilde{G}_{2, \delta_1, \delta_2, t_2, x_2}^p(0, \cdot) \right)(x_1),$$

where $\tilde{G}_{2, \delta_1, \delta_2, t_2, x_2}^p(t_1, x_1) := [\tilde{w}^2 + \tilde{z}^2]^{\frac{1}{2}}$ with \tilde{w}, \tilde{z} are even and odd extensions of w and z with respect to x_1 to \mathbb{R} , respectively, and $\widetilde{P_{t_1}^{[\lambda]}}(\tilde{G}_{2, \delta_1, \delta_2, t_2, x_2}^p(0, \cdot))(x_1)$ is the even extension of $P_{t_1}^{[\lambda]}(\tilde{G}_{2, \delta_1, \delta_2, t_2, x_2}^p(0, \cdot))(x_1)$ to \mathbb{R} with respect to x_1 .

We now show that $V_{\delta_1, \delta_2, t_2, x_2}$ satisfies (i)-(iv) of Lemma 5.1. In fact, since $V_{\delta_1, \delta_2, t_2, x_2}$ is an even function with respect to x_1 , we only need to consider $x_1 \in \mathbb{R}_+$. Since the assumption that f is restricted at infinity implies that $P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} f \in L^s(\mathbb{R}_\lambda)$ for all $t_1, t_2 \in \mathbb{R}_+$ and $s \in [p, \infty]$, by Lemma 2.9 (S_i), the uniform boundedness of $Q_t^{[\lambda]}$ on $L^2(\mathbb{R}_+, dm_\lambda)$ (see [MSt, p. 87]), we further obtain that for fixed $\delta_1, \delta_2, t_1, t_2 \in \mathbb{R}_+$,

$$\begin{aligned} (5.26) \quad & \iint_{\mathbb{R}_+ \times \mathbb{R}_+} G_2^2(\delta_1, \delta_2, t_1, t_2, x_1, x_2) d\mu_\lambda(x_1, x_2) \\ &= \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left[\left| P_{t_1}^{[\lambda]} Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) \right|^2 + \left| Q_{t_1}^{[\lambda]} Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) \right|^2 \right] d\mu_\lambda(x_1, x_2) \\ &\lesssim \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \left[P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f(x_1, x_2) \right]^2 d\mu_\lambda(x_1, x_2) < \infty. \end{aligned}$$

Observe that for all δ_1, δ_2, t_2 and almost every $x_1, x_2 \in (0, \infty)$,

$$[G_2(\delta_1, \delta_2, 0, t_2, x_1, x_2)]^2 = \left| Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2) \right|^2 + \left| R_{\Delta_\lambda, 1} Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2) \right|^2.$$

This fact together with Lemma 2.9 (S_i), the uniform boundedness of $Q_{t_2}^{[\lambda]}$ on $L^2(\mathbb{R}_+, dm_\lambda(x_2))$ and the boundedness of $R_{\Delta_\lambda, 1}$ on $L^2(\mathbb{R}_+, dm_\lambda(x_1))$ implies that

$$\begin{aligned} & \int_0^\infty \int_0^\infty \left[P_{t_1}^{[\lambda]} (G_2^p(\delta_1, \delta_2, 0, t_2, \cdot, x_2))(x_1) \right]^{2/p} d\mu_\lambda(x_1, x_2) \\ &\lesssim \int_0^\infty \int_0^\infty |G_2(\delta_1, \delta_2, 0, t_2, x_1, x_2)|^2 d\mu_\lambda(x_1, x_2) \\ &\lesssim \int_0^\infty \int_0^\infty \left[\left| Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2) \right|^2 + \left| R_{\Delta_\lambda, 1} Q_{t_2}^{[\lambda]} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right) (x_1, x_2) \right|^2 \right] d\mu_\lambda(x_1, x_2) \\ &\lesssim \int_0^\infty \int_0^\infty \left| P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f(x_1, x_2) \right|^2 d\mu_\lambda(x_1, x_2), \end{aligned}$$

which, together with (5.26), yields that for all δ_1, δ_2, t_2 and almost all x_2 ,

$$\sup_{0 < t_1 < \infty} \int_0^\infty |V_{\delta_1, \delta_2, t_2, x_2}(t_1, x_1)|^{2/p} dm_\lambda(x_1) \lesssim \int_0^\infty \left| P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f(x_1, x_2) \right|^2 dm_\lambda(x_1) < \infty.$$

Therefore, for fixed $\delta_1, \delta_2, t_2, x_2$, $V_{\delta_1, \delta_2, t_2, x_2}(t_1, x_1)$ satisfies the assumptions of Lemma 5.1 and hence (5.24) for G_2 follows from Lemma 5.1 immediately.

By an argument involving (5.24) and (5.25), we further see that for almost all x_1 and x_2 ,

$$G^p(\delta_1, \delta_2, t_1, t_2, x_1, x_2) \lesssim P_{t_2}^{[\lambda]} P_{t_1}^{[\lambda]} (G^p(\delta_1, \delta_2, 0, 0, \cdot, \cdot))(x_1, x_2).$$

Moreover, observe that

$$\begin{aligned} G(\delta_1, \delta_2, 0, 0, x_1, x_2) &\sim \left| P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]}(f)(x_1, x_2) \right| + \left| R_{\Delta_{\lambda}, 1} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right)(x_1, x_2) \right| \\ &\quad + \left| R_{\Delta_{\lambda}, 2} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right)(x_1, x_2) \right| + \left| R_{\Delta_{\lambda}, 1} R_{\Delta_{\lambda}, 2} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right)(x_1, x_2) \right|. \end{aligned}$$

Using these facts, (1.6) and Lemma 2.9 (S_i), we see that

$$\begin{aligned} (5.27) \quad &\int_0^\infty \int_0^\infty [G(\delta_1, \delta_2, t_1, t_2, x_1, x_2)]^p d\mu_\lambda(x_1, x_2) \\ &\lesssim \int_0^\infty \int_0^\infty P_{t_2}^{[\lambda]} P_{t_1}^{[\lambda]} ([G(\delta_1, \delta_2, 0, 0, \cdot, \cdot)]^p)(x_1, x_2) d\mu_\lambda(x_1, x_2) \\ &\lesssim \int_0^\infty \int_0^\infty |G(\delta_1, \delta_2, 0, 0, x_1, x_2)|^p d\mu_\lambda(x_1, x_2) \\ &\sim \int_0^\infty \int_0^\infty \left[\left| P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]}(f)(x_1, x_2) \right| + \left| R_{\Delta_{\lambda}, 1} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right)(x_1, x_2) \right| \right. \\ &\quad \left. + \left| R_{\Delta_{\lambda}, 1} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right)(x_1, x_2) \right| \right. \\ &\quad \left. + \left| R_{\Delta_{\lambda}, 1} R_{\Delta_{\lambda}, 2} \left(P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \right)(x_1, x_2) \right| \right]^p d\mu_\lambda(x_1, x_2) \lesssim 1, \end{aligned}$$

where the implicit constant is independent of t_1, t_2, δ_1 and δ_2 .

Observe that for each $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$G(\delta_1, \delta_2, t_1, t_2, x_1, x_2) \rightarrow F(t_1, t_2, x_1, x_2) \quad \text{as } \delta_1, \delta_2 \rightarrow 0,$$

Indeed, observe that $P_{\delta_1}^{[\lambda]} P_{\delta_2}^{[\lambda]} f \rightarrow f$ in $G(1, 1; 1, 1)'$ as $\delta_1, \delta_2 \rightarrow 0$. By these facts together with (5.27) and the Fatou lemma, we further have (5.23).

Let $q \in (\frac{2\lambda+1}{2\lambda+2}, p)$ and $r := p/q$. As in the proof of (5.17), we see that

$$(5.28) \quad F^q(\delta_1 + t_1, \delta_2 + t_2, x_1, x_2) \lesssim P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]} (F^q(\delta_1, \delta_2, \cdot, \cdot))(x_1, x_2).$$

Then by (5.23), there exist a subsequence $\{F^q(\delta_{1,k}, \delta_{2,j}, \cdot, \cdot)\}_{\delta_{1,k}, \delta_{2,j} > 0}$ of $\{F^q(\delta_1, \delta_2, \cdot, \cdot)\}_{\delta_1, \delta_2 > 0}$ and $h \in L^r(\mathbb{R}_\lambda)$ such that $\{F^q(\delta_{1,k}, \delta_{2,j}, \cdot, \cdot)\}_{\delta_{1,k}, \delta_{2,j} > 0}$ converges weakly to h in $L^r(\mathbb{R}_\lambda)$ as $k, j \rightarrow \infty$, which further implies that

$$\begin{aligned} (5.29) \quad \|h\|_{L^r(\mathbb{R}_\lambda)}^r &= \left\{ \sup_{\|g\|_{L^{r'}(\mathbb{R}_\lambda)} \leq 1} \lim_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} \left| \int_0^\infty \int_0^\infty g(x_1, x_2) F^q(\delta_{1,k}, \delta_{2,j}, x_1, x_2) d\mu_\lambda(x_1, x_2) \right| \right\}^r \\ &\leq \limsup_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} \|F^q(\delta_{1,k}, \delta_{2,j}, \cdot, \cdot)\|_{L^r(\mathbb{R}_\lambda)}^r \\ &\leq \limsup_{\substack{k \rightarrow \infty \\ j \rightarrow \infty}} \|F(\delta_{1,k}, \delta_{2,j}, \cdot, \cdot)\|_{L^p(\mathbb{R}_\lambda)}^p \\ &\lesssim 1. \end{aligned}$$

Moreover, by (5.28), we then have that for any $t_1, t_2, x_1, x_2 \in \mathbb{R}_+$,

$$F^q(t_1, t_2, x_1, x_2) \lesssim P_{t_1}^{[\lambda]} P_{t_2}^{[\lambda]}(h)(x_1, x_2).$$

Therefore,

$$[u^*(x_1, x_2)]^q \leq \sup_{\substack{t_1 > 0 \\ t_2 > 0}} F^q(t_1, t_2, x_1, x_2) \lesssim \mathcal{R}_P(h)(x_1, x_2).$$

By this together with the $L^r(\mathbb{R}_\lambda)$ -boundedness of \mathcal{R}_P and (5.29), we then have

$$\|u^*\|_{L^p(\mathbb{R}_\lambda)}^p = \|(u^*)^q\|_{L^r(\mathbb{R}_\lambda)}^r \lesssim \left\| \mathcal{R}_P(h) \right\|_{L^r(\mathbb{R}_\lambda)}^r \lesssim \|h\|_{L^r(\mathbb{R}_\lambda)}^r \lesssim 1.$$

This finishes the proof of Theorem 1.9. \square

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XUAN THINH DUONG, DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2019, AUSTRALIA
E-mail address: xuan.duong@mq.edu.au

JI LI, DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2019, AUSTRALIA
E-mail address: ji.li@mq.edu.au

BRETT D. WICK, DEPARTMENT OF MATHEMATICS, WASHINGTON UNIVERSITY – ST. LOUIS, ST. LOUIS, MO 63130-4899 USA
E-mail address: wick@math.wustl.edu

DONGYONG YANG, SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN 361005, CHINA
E-mail address: dyyang@xmu.edu.cn